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FUNCTORIALITY OF REAL ANALYTIC TORSION FORMS

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ABSTRACT

In this paper, we prove the functoriality of the analytic torsion forms of Bismut and Lott [BLo] with respect to the composition of two submersions.

0 Introduction

In [BLo], Bismut and Lott extended the famous Ray–Singer analytic torsion [RS1] from an invariant of a smooth manifold to the family case. Namely, they introduced a real analytic torsion form for a smooth fibration. One of the significant facts is that the real analytic torsion form enters in a differential form version of a $\mathcal{C}^\infty$-analog of the Riemann–Roch–Grothendieck theorem for holomorphic submersions.

The purpose of this paper is to prove the functoriality of the real analytic torsion form with respect to the composition of two submersions. Let us state some of our results in detail.

Let W, V, S be smooth manifolds. Let $\pi_1: W \to V$, $\pi_2: V \to S$ be smooth fibrations with compact fibre X, Y. Then $\pi_3 = \pi_2 \circ \pi_1: W \to S$ is a smooth fibration with compact fiber Z of dimension n. Let TX, TY, TZ be the relative tangent bundles. Let (F, ∇^F) be a flat complex vector bundle over W. Let h^F be a Hermitian metric on F. Then we have the diagram of smooth fibrations:

$$\begin{tikzcd}
X \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"'] & W \arrow[d, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
& Y \arrow[r] & V \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

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Let $H^\bullet(X,F_{|X})=\bigoplus_{i=0}^{\dim X} H^i(X,F_{|X})$, $H^\bullet(Z,F_{|Z})$, $H^\bullet(Y,H^\bullet(X,F_{|X}))$ be the $\mathbf{Z}$-graded vector bundles over V,S,S whose fibers over $a\in V$, $s\in S$ are the cohomologies $H^\bullet(X_a,F_{|X_a})$, $H^\bullet(Z_s,F_{|Z_s})$, $H^\bullet(Y_s,H^\bullet(X,F_{|X}))$ of the sheaf of locally flat sections of F, F, $H(X,F_{|X})$ on X_a, Z_s, Y_s. Let $\nabla^{H(X,F_{|X})}$ be the canonical flat connection on $H^\bullet(X,F_{|X})$.

Let T_1^HW,T_2^HV,T_3^HW be the sub-bundles of TW,TV,TW which are complements of TX,TY,TZ. Let g^{TZ},g^{TX},g^{TY} be metrics on TZ,TX,TY.

Let $(\Omega(X,F_{|X}),d^X)$ be the de Rham complex of smooth sections of $\Lambda(T^*X)\otimes F$ over X. By Hodge theory, we can identify $H^\bullet(X,F_{|X})$ with the corresponding harmonic elements in the de Rham complex $(\Omega(X,F_{|X}),d^X)$. Let $h^{H(X,F_{|X})}$ be the corresponding L^2-metric on $H^\bullet(X,F_{|X})$ with respect to g^{TX},h^F. In the same way, we note $h^{H(Z,F_{|Z})}$, $h^{H(Y,H(X,F_{|X}))}$ the corresponding L^2 metrics on $H^\bullet(Z,F_{|Z})$, $H^\bullet(Y,H^\bullet(X,F_{|X}))$ induced by g^{TZ},h^F and $g^{TY},h^{H(X,F_{|X})}$.

Let $\nabla^{TX},\nabla^{TY},\nabla^{TZ}$ be the connections on TX,TY,TZ defined in [B1, Definition 1.6]. Let $T^HZ=T_1^HW\cap TZ$. Let $\pi_1^*\nabla^{TY}$ be the connection on T^HZ induced by ∇^{TY}. Then ${}^0\nabla^{TZ}=\pi_1^*\nabla^{TY}\oplus\nabla^{TX}$ is a connection on $TZ=T^HZ\oplus TX$. Let $e(TX,\nabla^{TX}),e(TY,\nabla^{TY})$, $e(TZ,\nabla^{TZ})$, $e(TZ,{}^0\nabla^{TZ})$ be the associated representatives of the Euler class of TX,TY, TZ, TZ in Chern–Weil theory. Let $\widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})$ be the Chern–Simons $n-1$ forms on W with values in $o(TZ)$, the orientation bundle of TZ, such that

$$d\widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})=e(TZ,{}^0\nabla^{TZ})-e(TZ,\nabla^{TZ}). \tag{0.1}$$

Let $f(\nabla^F,h^F)$ be the closed odd forms on W defined in (1.28), which are the analogue of the Chern character on the flat vector bundle F.

Let Q^S be the vector space of real even forms on S. Let $Q^{S,0}$ be the vector space of real exact even forms on S.

Let $\mathcal{T}(T_1^HW,g^{TX},h^F),\mathcal{T}(T_2^HV,g^{TY},h^{H(X,F_{|X})})$, $\mathcal{T}(T_3^HW,g^{TZ},h^F)$ be the analytic torsion forms corresponding to π_1,π_2,π_3 defined in [BLo, Definition 3.22]. The form $\mathcal{T}(T_1^HW,g^{TX},h^F)$ satisfies the following equation:

(0.2)
$$d\mathcal{T}(T_1^HW,g^{TX},h^F)=\int_X e(TX,\nabla^{TX})f(\nabla^F,h^F)-f(\nabla^{H(X,F_{|X})},h^{H(X,F_{|X})}).$$

For $s\in S$, the Leray spectral sequence $(E_{r,s},d_{r,s})$ $(r\geq 2)$ [Grot] with respect to $\pi_1\colon Z_s\to Y_s$, verifies $E_2=H(Y,H(X,F_{|X}))$. Let h^{E_2} be the metric on E_2 induced by $h^{H(Y,H(X,F_{|X}))}$.

By Proposition 3.1, we know that $(E_r,d_r),(r\geq 2)$ is a flat complex of vector bundles on S. And the de Rham complex $\Omega(Z_s,F_{|Z_s})$, provided with a suitable filtration, calculates the Leray spectral sequence. By using these two facts, in Definition 3.2, we define the form $T(E_2, H(Z, F_{|Z}), h^{E_2}, h^{H(Z,F_{|Z})})$ on S such that

$$\tag{0.3} dT(E_2, H(Z, F_{|Z}), h^{E_2}, h^{H(Z,F_{|Z})}) = f(\nabla^{E_2}, h^{E_2}) - f(\nabla^{H(Z,F_{|Z})}, h^{H(Z,F_{|Z})}).$$

The purpose of this paper is to establish the following result, which we state as Theorem 3.1,

THEOREM 0.1: *The following identity holds in $Q^S/Q^{S,0}$,*

$$\tag{0.4} \begin{aligned} \mathcal{T}(T_3^H W, g^{TZ}, h^F) = &\int_Y e(TY, \nabla^{TY}) \mathcal{T}(T_1^H W, g^{TX}, h^F) \\ &+ \mathcal{T}(T_2^H V, g^{TY}, h^{H(X,F_{|X})}) \\ &+ T(E_2, H(Z, F_{|Z}), h^{E_2}, h^{H(Z,F_{|Z})}) \\ &- \int_Z \widetilde{e}(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) f(\nabla^F, h^F). \end{aligned}$$

In [Lo], Lott defined a secondary K-group for flat complex Hermitian vector bundles on a $\mathcal{C}^\infty$ manifold. Lott defined also the direct image (secondary index) in his secondary K-group for a $\mathcal{C}^\infty$ fibration with compact fibre, and the real analytic torsion form is one part of his secondary index. We can consider it as a $\mathcal{C}^\infty$ analogue of Gillet–Soulé's arithmetic K-Theory in Arakelov goemetry. In [Bu], Bunke shows that Theorem 0.1 actually implies the functoriality of Lott's secondary indices [Lo].

Assume now that S is a point. Then we have a submersion $\pi_1\colon Z \to Y$ of compact manifolds with fibre X. Let

$$\tag{0.5} \begin{aligned} \lambda(F) &= \bigotimes_{i=0}^{\dim Z} (\det H^i(Z, F))^{(-1)^i}, \\ \lambda(H^\bullet(X, F_{|X})) &= \bigotimes_{i,j=0}^{\dim Z} (\det H^i(Y, H^j(X, F_{|X})))^{(-1)^{i+j}} \end{aligned}$$

be the determinant of the cohomologies of F, $H^\bullet(X, F_{|X})$. By [KM], we have a canonical nonzero section $\sigma \in \lambda^{-1}(H(X, F_{|X})) \otimes \lambda(F)$.

Let $\|\ \|_{\lambda(H(X,F_{|X}))}$, $\|\ \|_{\lambda(F)}$ be the Ray–Singer metrics on $\lambda(H(X, F_{|X}))$, $\lambda(F)$ associated to the metrics $g^{TY}, h^{H(X,F_{|X})}$, and g^{TZ}, h^F [BZ, Definition 2.2]. Let $\|\ \|_{\lambda^{-1}(H(X,F_{|X}))\otimes\lambda(F)}$ be the corresponding Ray–Singer metric on $\lambda^{-1}(H(X, F_{|X})) \otimes \lambda(F)$. Let $T(X, h^F)$ be the Ray-Singer analytic torsion [RS1, Definition 1.6] on the fibre X associated to the metrics g^{TX}, h^F.

By [BLo, Theorems 2.25 and 3.29], and (1.28), we can reformulate Theorem 0.1,

$$\begin{aligned}(0.6)\quad \log(\|\sigma\|_{\lambda^{-1}(H(X,F_{|X}))\otimes\lambda(F)}) = &\int_Y e(TY,\nabla^{TY})\log T(X,h^F)\\ &-\frac{1}{2}\int_Z \widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\,\mathrm{Tr}[(h^F)^{-1}\nabla^F h^F].\end{aligned}$$

If Z is oriented, odd dimensional, and h^F is a flat metric, let $g^{TZ}_\varepsilon = \varepsilon^2 g^{TZ} + \pi^* g^{TY}$. Let $T_\varepsilon(Z,h^F)$ be the Ray–Singer analytic torsion associated to g^{TZ}_ε. In [D1], [DM], Dai and Melrose have calculated the asymptotics of $T_\varepsilon(Z,h^F)$ as $\varepsilon \to 0$. In [LST], Lück, Schick and Thielmann have generalized it to the case that F is unimodular, and that Z is odd or even. In fact, by using [BZ, Theorems 0.1, 0.2], [Mü], they show their main result [LST, Theorem 0.2] follows from the corresponding result on Reidemeister torsion which is essentially a problem of finite dimensional linear algebra.

So the equation (0.6) extends the results of [DM], [LST], to the general case, where F is not necessarily unimodular. Furthermore, we do not use the result [BZ, Theorem 0.2]. Dai told me that their method also works in this case.

This paper is organized as follows: In Section 1, we recall the construction of the analytic torsion forms of Bismut and Lott [BLo]. In Section 2, we prove that the de Rham complex, provided with a suitable filtration, calculates the Leray spectral sequence. We also give a derivation of Dai's result on the small eigenvalues [D]. In Section 3, we state our main result, Theorem 3.1. In Section 4, we state seven intermediate results, whose proofs are delayed to Sections 5–9. We then prove Theorem 3.1. Sections 5–9 are devoted to the proof of the intermediate results which were alluded to before.

This paper is a revised version of [Ma3].

In the whole paper, if A is a $\mathbf{Z}_2$-graded algebra, and if $a,b \in A$, then we will note $[a,b]$ as the supercommutator of a,b. And if $J \in \mathrm{End}(A)$, we denote $\mathrm{Tr}_s(J)$ as the supertrace of J [BeGeV, §1.3].

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1. Analytic torsion forms

In this Section, we recall the construction of analytic torsion forms [BLo].

This Section is organized as follows. In Section 1.1, we introduce the superconnection of Bismut-Lott. In Section 1.2, we recall the construction of the flat connection on the cohomology bundle of fibers. In Section 1.3, we construct the analytic torsion forms.

1.1. Superconnection of Bismut-Lott. Let $\pi\colon W \to S$ be a smooth fiber bundle with compact fiber Z of dimension n. Let TZ be the vertical tangent bundle of the fiber bundle, and let T^*Z be its dual bundle. Let F be a flat complex vector bundle on W and let ∇^F denote its flat connection.

Let T^HW be a sub-bundle of TW such that

$$TW = T^HW \oplus TZ. \tag{1.1}$$

Let P^{TZ} denote the projection from TW to TZ. If $U \in TS$, let U^H be the lift of U in T^HW, so that $\pi_* U^H = U$.

Let $E = \bigoplus_{i=0}^n E^i$ be the smooth infinite-dimensional $\mathbf{Z}$-graded vector bundle over S whose fiber over $s \in S$ is $\mathcal{C}^\infty(Z_s, (\Lambda(T^*Z) \otimes F)_{|Z_s})$. That is

$$\mathcal{C}^\infty(S, E^i) = \mathcal{C}^\infty(W, \Lambda^i(T^*Z) \otimes F). \tag{1.2}$$

Definition 1.1: For $s \in \mathcal{C}^\infty(S; E)$ and U a vector field on S, then the Lie differential L_{U^H} acts on $\mathcal{C}^\infty(S, E)$. Set

$$\nabla^E_U s = L_{U^H} s. \tag{1.3}$$

Then ∇^E is a connection on E which preserves the $\mathbf{Z}$-grading.

If U_1, U_2 are vector fields on S, put

$$T(U_1, U_2) = -P^{TZ}[U_1^H, U_2^H] \in \mathcal{C}^\infty(W, TZ). \tag{1.4}$$

We denote $i_T \in \Omega^2(S, \mathrm{Hom}(E^\bullet, E^{\bullet-1}))$ to be the 2-form on S which, to vector fields U_1, U_2 on S, assigns the operation of interior multiplication by $T(U_1, U_2)$ on E. Let d^Z be exterior differentiation along fibers. We consider d^Z to be an element of $\mathcal{C}^\infty(S, \mathrm{Hom}(E^\bullet, E^{\bullet+1}))$. The exterior differentiation operator d^W, acting on $\mathcal{C}^\infty(W, \Lambda(T^*W) \otimes F)$, has degree 1 and satisfies $(d^W)^2 = 0$. By [BLo, Proposition 3.4], we have

$$d^W = d^Z + \nabla^E + i_T. \tag{1.5}$$

So d^W is a flat superconnection of total degree 1 on E. We have

$$(d^Z)^2 = 0, \quad [\nabla^E, d^Z] = 0. \tag{1.6}$$

Let g^{TZ} be a metric on TZ. Let h^F be a Hermitian metric on F. Let ∇^{F*} be the adjoint of ∇^F with respect to h^F.

Definition 1.2: Let $\omega(F, h^F)$ be the 1-form on W taking values in self-adjoint endomorphisms of F,

$$\omega(F, h^F) = (h^F)^{-1}\nabla^F h^F. \tag{1.7}$$

Let $\nabla^{F,u}$ be the connection on F,

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega(F, h^F) = \frac{1}{2}(\nabla^F + \nabla^{F*}). \tag{1.8}$$

Let $o(TZ)$ be the orientation bundle of TZ, a flat real line bundle on W. Let dv_Z be the Riemannian volume form on fibers Z associated to the metric g^{TZ}. (Here dv_Z is viewed as a section of $\Lambda^n(T^*Z) \otimes o(TZ)$.) Let $\langle\ \rangle_{\Lambda(T^*Z)\otimes F}$ be the metric on $\Lambda(T^*Z) \otimes F$ induced by g^{TZ}, h^F. Let $*$ be the fiberwise Hodge duality operator associated to g^{TZ}. Then E acquires a Hermitian metric h^E such that for $\alpha, \alpha' \in \mathcal{C}^\infty(S, E)$ and $s \in S$,

$$\langle \alpha, \alpha' \rangle_{h^E} = \int_{Z_s} \langle \alpha \wedge *\alpha' \rangle_F = \int_{Z_s} \langle \alpha, \alpha' \rangle_{\Lambda(T^*Z)\otimes F}\, dv_{Z_s}. \tag{1.9}$$

Let ∇^{E*}, d^{Z*}, $(d^W)^*$, $(i_T)^*$ be the formal adjoint of ∇^E, d^Z, d^W, i_T with respect to the scalar product $\langle , \rangle_{h^E}$. Set

$$\begin{aligned} D^Z &= d^Z + d^{Z*}, \qquad \nabla^{E,u} = \frac{1}{2}(\nabla^E + \nabla^{E*}), \\ \omega(E, h^E) &= \nabla^{E*} - \nabla^E. \end{aligned} \tag{1.10}$$

Let N_Z be the number operator of E, i.e., acts by multiplication by k on $\mathcal{C}^\infty(W, \Lambda^k(T^*Z) \otimes F)$. For $u > 0$, set

$$\begin{aligned} C'_u &= u^{N_Z/2} d^W u^{-N_Z/2}, \quad C''_u = u^{-N_Z/2}(d^W)^* u^{N_Z/2}, \\ C_u &= \frac{1}{2}(C'_u + C''_u), \quad D_u = \frac{1}{2}(C''_u - C'_u). \end{aligned} \tag{1.11}$$

Then C''_u is the adjoint of C'_u with respect to h^E; C_u is a superconnection and D_u is an odd element of $\Omega(S, \mathrm{End}(E))$, and

$$C_u^2 = -D_u^2. \tag{1.12}$$

In calculations we will sometimes assume that S has a Riemannian metric g^{TS} and W has the Riemannian metric $g^{TW} = g^{TZ} \oplus \pi^* g^{TS}$, although all final results will be independent of g^{TS}. Let ∇^{TW}, ∇^{TS} denote the corresponding Levi-Civita connections on W, S. Put $\nabla^{TZ} = P^{TZ}\nabla^{TW}$, a connection on TZ. As shown in [B1, Theorem 1.9], ∇^{TZ} is independent of the choice of g^{TS}. Then ${}^0\nabla = \nabla^{TZ} \oplus \pi_1^* \nabla^{TS}$ is also a connection on TW. Let $S = \nabla^{TW} - {}^0\nabla$. By [B1, Theorem 1.9], $\langle S(.).,.\rangle_{g^{TW}}$ is a tensor independent of g^{TS}.

Let g_α be a base of TS; set g^α the dual base of T^*S. Let e_i be an orthonormal base of (TZ, g^{TZ}). We define a horizontal 1-form k on W by

$$k(g_\alpha) = -\sum_i \langle S(e_i)e_i, g_\alpha\rangle . \tag{1.13}$$

For $X \in TZ$, let $X^* \in T^*Z$ correspond to X by the metric g^{TZ}. Set

$$c(X) = X^* \wedge -i_X, \quad \widehat{c}(X) = X^* \wedge +i_X. \tag{1.14}$$

Set

$$c(T) = \frac{1}{2}\sum_{\alpha,\beta} g^\alpha \wedge g^\beta c(T(g_\alpha, g_\beta)). \tag{1.15}$$

Let $\nabla^{\Lambda(T^*Z)}$ be the connection on $\Lambda(T^*Z)$ induced by ∇^{TZ}. Let $\nabla^{TZ\otimes F,u}$ be the connection on $\Lambda(T^*Z)\odot F$ induced by $\nabla^{\Lambda(T^*Z)}$, $\nabla^{F,u}$. Then by [BLo, (3.36), (3.37)],

$$\begin{aligned} D^Z &= c(e_j)\nabla^{TZ\otimes F,u}_{e_j} - \frac{1}{2}\widehat{c}(e_j)\omega(F,h^F)(e_j), \\ \nabla^{E,u} &= g^\alpha\left(\nabla^{TZ\otimes F,u}_{g_\alpha} + \frac{1}{2}k(g_\alpha)\right), \\ \omega(E,h^E) &= g^\alpha\left(\langle S(g_\alpha)e_i, e_j\rangle\, c(e_i)\widehat{c}(e_j) + \omega(F,h^F)(g_\alpha)\right). \end{aligned} \tag{1.16}$$

By [BLo, Proposition 3.9], we get

$$C_u = \frac{\sqrt{u}}{2}D^Z + \nabla^{E,u} - \frac{1}{2\sqrt{u}}c(T). \tag{1.17}$$

Remark that in [Zh, §2c)], Zhang observed that we can obtain this Bismut–Lott superconnection from his sub-signature operator in the same way as the Bismut superconnection is obtained from the Dirac operator.

Let R^{TZ} be the curvature of ∇^{TZ}. Set

$$\widehat{R}^{TZ} = \frac{1}{4}\left\langle e_i, R^{TZ}e_j\right\rangle_{g^{TZ}} \widehat{c}(e_i)\widehat{c}(e_j). \tag{1.18}$$

1.2. The flat connection on the cohomology bundle of the fibers. Let $H^\bullet(Z,F_{|Z})=\bigoplus_{i=0}^{\dim Z}H^i(Z,F_{|Z})$ be the $\mathbf{Z}$-graded vector bundle over S whose fiber over $s\in S$ is the cohomology $H(Z_s,F_{|Z_s})$ of the sheaf of locally flat sections of F on Z_s. By [BLo, §3 (f)], the flat superconnection d^W induces a canonical flat connection $\nabla^{H(Z,F_{|Z})}$ on $H^\bullet(Z,F_{|Z})$ which preserves the $\mathbf{Z}$-grading. The connection $\nabla^{H(Z,F_{|Z})}$ does not depend on the choice of T^HM, and is the canonical flat connection on $H^\bullet(Z,F_{|Z})$.

By Hodge theory, there is an isomorphism

$$H^\bullet(Z_s,F_{|Z_s})\simeq \operatorname{Ker}(D^Z_s). \tag{1.19}$$

Then there is an isomorphism of smooth $\mathbf{Z}$-graded vector bundles on S

$$H^\bullet(Z,F_{|Z})\simeq \operatorname{Ker}(D^Z). \tag{1.20}$$

Clearly $\operatorname{Ker}(D^Z)$ inherits a metric from the scalar product $\langle\ \ \rangle_{h^E}$. Let $h^{H(Z,F_{|Z})}$ be the corresponding metric on $H^\bullet(Z,F_{|Z})$.

Let P be the orthogonal projection operator from E on $\operatorname{Ker}(D^Z)$ with respect to the Hermitian product (1.9). Set $P^\perp=1-P$. Let $(\nabla^{H(Z,F_{|Z})})^*$ be the adjoint of $\nabla^{H(Z,F_{|Z})}$ with respect to the Hermitian metric $h^{H(Z,F_{|Z})}$. Put

$$\nabla^{H(Z,F_{|Z}),u}=\frac{1}{2}\left(\nabla^{H(Z,F_{|Z})}+(\nabla^{H(Z,F_{|Z})})^*\right), \tag{1.21}$$

a Hermitian connection on $H(Z,F_{|Z})$.

The following result is established in [BLo, Proposition 3.14].

Proposition 1.1: *The following identities hold:*

$$\begin{aligned}&\nabla^{H(Z,F_{|Z})}=P\nabla^E,\quad \left(\nabla^{H(Z,F_{|Z})}\right)^*=P\nabla^{E*},\\ &\omega\left(H(Z,F_{|Z}),h^{H(Z,F_{|Z})}\right)=P\omega(E,h^E)P.\end{aligned} \tag{1.22}$$

1.3. Analytic torsion forms. Let $\operatorname{Pf}\colon so(m)\to\mathbf{R}$ denote the Pfaffian. Set

$$e(TZ,\nabla^{TZ})=\operatorname{Pf}\Big[\frac{R^{TZ}}{2\pi}\Big]. \tag{1.23}$$

Then $e(TZ,\nabla^{TZ})$ is an $o(TZ)$ value closed n-form on W which represents the Euler class $e(TZ)$ of TZ, lying in $H^n(W,o(TZ))$ [BZ, (3.17)]. Of course, $e(TZ,\nabla^{TZ})=0$, if n is odd. Put

$$\chi(Z)=\sum_{i=0}^n(-1)^i\mathrm{rk}H^i(Z,\mathbf{R}),\quad \chi'(Z,F)=\sum_{i=0}^n(-1)^i i\,\mathrm{rk}H^i(Z,F_{|Z}). \tag{1.24}$$

Then $\chi(Z)$ is the Euler characteristic number of TZ. And $\chi(Z)$, $\chi'(Z,F)$ are locally constant functions on S.

Let $\Omega(S)$, $\Omega(W)$ denote the space of smooth sections of $\Lambda(T^*S)$, $\Lambda(T^*W)$. Let φ: $\Omega(W) \to \Omega(W)$ (resp. $\Omega(S) \to \Omega(S)$) be the linear map such that for all homogeneous $\omega \in \Omega(W)$ (resp. $\Omega(S)$),

$$\varphi\omega = (2\pi i)^{-(\deg \omega)/2}\omega. \tag{1.25}$$

Definition 1.3: Let Q^W be the vector space of real even forms on W. Let $Q^{W,0}$ be the vector space of real exact even forms on W.

For $a \in \mathbf{C}$, put

$$f(a) = a\exp(a^2), \quad g(a) = (1-2a)\exp(-a). \tag{1.26}$$

We have

$$f'(a) = (1+2a^2)\exp(a^2). \tag{1.27}$$

Put

$$f(\nabla^F, h^F) = (2i\pi)^{1/2}\varphi \operatorname{Tr}[f(\frac{1}{2}\omega(F,h^F))] \in \Omega(W). \tag{1.28}$$

Then $f(\nabla^F, h^F)$ is closed and its de Rham cohomology class is independent of h^F.

For any $u > 0$, the operator D_u is a fiberwise-elliptic differential operator. Then $f(D_u)$ is a fiberwise trace class operator. For $u > 0$, put

$$\begin{aligned} f(C'_u, h^E) &= (2i\pi)^{1/2}\varphi \operatorname{Tr}_s[f(D_u)], \\ f^\wedge(C'_u, h^E) &= \varphi \operatorname{Tr}_s\left[\frac{N_Z}{2} f'(D_u)\right], \\ f(\nabla^{H(Z,F_{|Z})}, h^{H(Z,F_{|Z})}) &= \sum_{q=0}^{\dim Z} (-1)^q f(\nabla^{H^q(Z,F_{|Z})}, h^{H(Z,F_{|Z})}). \end{aligned} \tag{1.29}$$

The following results are proved in [BLo, Theorem 3.16],

THEOREM 1.1: *For any $u > 0$, the form $f(C'_u, h^E)$ is real, odd, and closed. Its de Rham cohomology class is independent of u, T^HW, g^{TZ} and h^F. As $u \to 0$,*

$$f(C'_u, h^E) = \begin{cases} \int_Z e(TZ, \nabla^{TZ}) f(\nabla^F, h^F) + O(u) & \text{if } \dim Z \text{ is even,} \\ O(\sqrt{u}) & \text{if } \dim Z \text{ is odd.} \end{cases} \tag{1.30}$$

As $u \to +\infty$

$$f(C'_u, h^E) = f(\nabla^{H(Z,F_{|Z})}, h^{H(Z,F_{|Z})}) + O(1/\sqrt{u}). \tag{1.31}$$

The following results are proved in [BLo, Theorems 3.20 and 3.21], THEOREM 1.2: *For any $u>0$, the form $f^\wedge(C'_u,h^E)$ is real and even. Moreover,*

$$\frac{\partial}{\partial u} f(C'_u,h^E)=\frac{1}{u}df^\wedge(C'_u,h^E). \tag{1.32}$$

As $u\to 0$,

$$f^\wedge(C'_u,h^E)=\begin{cases}\frac{1}{4}\dim Z\,\mathrm{rk}(F)\chi(Z)+O(u) & \text{if } \dim Z \text{ is even},\\ O(\sqrt{u}) & \text{if } \dim Z \text{ is odd}.\end{cases} \tag{1.33}$$

As $u\to+\infty$

$$f^\wedge(C'_u,h^E)=\frac{1}{2}\chi'(Z,F)+O(1/\sqrt{u}). \tag{1.34}$$

Definition 1.4: The analytic torsion form $\mathcal{T}(T^HW,g^{TZ},h^F)\in\Omega(S)$ is given by

$$\begin{aligned}\mathcal{T}(T^HW,g^{TZ},h^F)=-\int_0^{+\infty}\Big[&f^\wedge(C'_u,h^E)-\frac{1}{2}\chi'(Z,F)f'(0)\\ &-\Big(\frac{1}{4}\dim Z\,\mathrm{rk}(F)\chi(Z)-\frac{1}{2}\chi'(Z,F)\Big)f'\Big(\frac{i\sqrt{u}}{2}\Big)\Big]\frac{du}{u}.\end{aligned} \tag{1.35}$$

The following results are proved in [BLo, Theorem 3.23].

THEOREM 1.3: *The form $\mathcal{T}(T^HW,g^{TZ},h^F)$ is even and real. Moreover,*

$$d\mathcal{T}(T^HW,g^{TZ},h^F)=\int_Z e(TZ,\nabla^{TZ})f(\nabla^F,h^F)-f(\nabla^{H(Z,F|_Z)},h^{H(Z,F|_Z)}). \tag{1.36}$$

2. Leray spectral sequence

This Section is organized as follows. In Section 2.1, we prove that the de Rham complex, provided with a suitable filtration, calculates the Leray spectral sequence. In Section 2.2, by following [BerB, §6], we give a derivation of Dai's result on the small eigenvalues [D].

2.1. DE RHAM COMPLEX. Let $\pi_1\colon Z\to Y$ be a fibration of compact manifolds with compact fibre X. Let F be a flat complex vector bundle on Z.

As in [BerB, (1.3)], [GrH, p. 464], let

$$\Lambda(T^*Z)=F^0(\Lambda(T^*Z))\supset F^1(\Lambda(T^*Z))\supset\cdots\supset F^{\dim Y+1}(\Lambda(T^*Z))=\{0\} \tag{2.1}$$

be the standard filtration of $\Lambda(T^*Z)$. In fact $F^p\Lambda^q(T^*Z)$ are the forms which can be written as a finite sum of forms of the shape $\omega\wedge\pi^*\eta$ for $\omega\in\Lambda^{q-k}(T^*Z)$, $\eta\in\Lambda^k(T^*Y)$ for some $k\geq p$. The filtration (2.1) induces a corresponding filtration of the complex $(\Omega(Z,F),d^F)$ such that $F^p\Omega(Z,F)=\mathcal{C}^\infty(Z,F^p\Lambda(T^*Z)\otimes F)$. We also get a corresponding filtration on $H^\bullet(Z,F)$. Set

$$\mathrm{Gr}^p H^\bullet(Z,F)=\frac{F^pH^\bullet(Z,F)}{F^{p+1}H^\bullet(Z,F)},\quad \mathrm{Gr}^\bullet H^\bullet(Z,F)=\bigoplus_{p=0}^{\dim Y}\mathrm{Gr}^p H^\bullet(Z,F). \tag{2.2}$$

For $b\in Y$, let $(\Omega(X_b,F_{|X_b}),d^X)$ be the relative de Rham complex of smooth sections of $(\Lambda(T^*X)\otimes F)_{|X_b}$. The $\Omega(X_b,F_{|X_b})$'s will be considered as the fibers of an infinite dimensional vector bundle over Y, whose smooth sections are identified with the smooth sections of $\Lambda(T^*X)\otimes F$ on Z. Let $\Omega^\bullet(Y,\Omega^\bullet(X,F_{|X}))$ be the vector space of smooth sections of $\Lambda(T^*Y)\otimes\Omega^\bullet(X,F_{|X})$ on Y. Then we have

$$\Omega^\bullet(Y,\Omega^\bullet(X,F_{|X}))\simeq\Omega^\bullet(Z,F_{|Z}). \tag{2.3}$$

Let (E_r,d_r) be the spectral sequence associated to the filtration (2.1) on the filtered complex $(\Omega(Z,F),d^F)$ [GrH, §3.5]. Then, as in [BerB, §1 (a)], we get

$$\begin{aligned}(E_0^{\bullet,\bullet},d_0)&=(\Omega^\bullet(Y,\Omega^\bullet(X,F_{|X})),d^X),\\ (E_1^{\bullet,\bullet},d_1)&=(\Omega^\bullet(Y,H(X,F_{|X})),d^Y),\\ E_2^{p,q}&=H^p(Y,H^q(X,F_{|X})).\end{aligned} \tag{2.4}$$

And E_2 is a finite dimensional $\mathbf{Z}$-graded vector space. More generally, for any $r\geq 0$, E_{r+1} is the cohomology of the complex (E_r,d_r). And for $r>\dim Z$,

$$(E_r^{\bullet,\bullet},d_r)=(\mathrm{Gr}^\bullet H^\bullet(Z,F),0). \tag{2.5}$$

By [Grot, Theorem 3.7.3], there is a functor of the Leray spectral sequence associated to the fibration $\pi_1\colon Z\to Y$.

THEOREM 2.1: *(E_r,d_r) $(r\geq 2)$ calculates the Leray spectral sequence.*

Proof: Let $\mathcal{D}_\infty(Z)$ be the sheaf of $\mathcal{C}^\infty$ sections of $\Lambda(T^*Z)$ on Z. Then $\mathcal{D}_\infty$ is $\mathbf{R}$-flat. Following the proof of [Ma2, §2(a)], by exchanging anti-holomorphic cotangent bundle by cotangent bundle in our context, we get Theorem 2.1. ∎

2.2. SMALL EIGENVALUES. In this part, we fix the sub-bundle T^HZ of TZ as in (1.1). Let T_1 be the tensor defined in (1.4) for $\pi_1\colon Z\to Y$. Let N_X be the number operator on $\Omega(X,F_{|X})$. Let $\nabla^{\Omega(X,F_{|X})}$ be the connection on $\Omega(X,F_{|X})$ as in (1.3). Set

$$d^H=\nabla^{\Omega(X,F_{|X})},\quad d_T^Z=T^{N_X}d^ZT^{-N_X}. \tag{2.6}$$

Let g^{TX}, g^{TY} be metrics on TX, TY. Let $\langle\ \ \rangle_1$ be the product on $\Omega(Z,F)$ defined in (1.9) with respect to $g^{TX} \oplus \pi_1^* g^{TY}$, h^F on TZ, F. Let d_T^{Z*}, d^{Z*}, d^{X*}, $d^{H*}, i_{T_1}^*$ be the formal adjoints of d_T^Z, d^Z, d^X , d^H, i_{T_1} with respect to $\langle\ \ \rangle_1$ on $\Omega(Z,F)$. Let $\{f_l\}$ be an orthonormal basis of (TY, g^{TY}), and $\{f^l\}$ be its dual basis. Then

$$i_{T_1} = \frac{1}{2}\sum_{k,l} f^k \wedge f^l \wedge i_{T_1(f_k,f_l)}, \quad i_{T_1}^* = \frac{1}{2}\sum_{k,l} i_{f_k} \wedge i_{f_l} \wedge (T_1(f_k,f_l))^*. \tag{2.7}$$

Definition 2.1: Set

$$D^X = d^{X*} + d^X, \quad D^H = d^{H*} + d^H. \tag{2.8}$$

Now, we define a sequence of Hermitian subspaces E'_r of $E = \Omega(Z,F)$, $E = E'_0 \supset E'_1 \supset \cdots \supset E'_r \supset \cdots$ such that

$$(E'_r, d'_r) \simeq (E_r, d_r). \tag{2.9}$$

By (2.3) and (2.4), set

$$E'_0 = E_0 = E. \tag{2.10}$$

Suppose that we have constructed $E'_{r'}, (r' \le r)$. As $E'_r \simeq E_r$, the operator d_r acts on E'_r. Let d_r^* be the adjoint of d_r with respect to the metric on E'_r. Set

$$D_r = d_r + d_r^*, \quad E'_{r+1} = \operatorname{Ker} D_r. \tag{2.11}$$

Then $E'_{r+1} \subset E'_r$, and E'_{r+1} inherits a Hermitian product on E'_r. Let p_r be the orthogonal projection from E on E'_r. By Hodge theory,

$$E'_{r+1} \simeq E_{r+1}. \tag{2.12}$$

The following result first appeared in [B3]. This is an analogue of [BerB, Theorem 6.1].

PROPOSITION 2.1: *For $r \in \mathbf{N}$, E'_r splits as an orthogonal direct sum $E'_r = \oplus_{p,q} E'^{p,q}_r$, with $E'^{p,q}_r \subset E_0^{p,q}$, so that under the identification $(E'_r, d_r) \simeq (E_r, d_r)$, we have $E'^{p,q}_r \simeq E_r^{p,q}$. For any $r \in \mathbf{N}$*

(2.13)

$$\begin{aligned} E'_r = \{ & s_0 \in \Omega(Z,F), \text{ there exist } s_1, \dots, s_{r-1} \in \Omega(Z,F), \text{ such that} \\ & D^X s_0 = 0, D^H s_0 + D^X s_1 = 0, (i_{T_1} + i_{T_1}^*) s_0 + D^H s_1 + D^Z s_2 = 0, \\ & \dots, (i_{T_1} + i_{T_1}^*) s_{r-3} + D^H s_{r-2} + D^Z s_{r-1} = 0 \}. \end{aligned}$$

If $s_0 \in E'_r$, *then*

$$D_r s_0 = p_r(D^H s_{r-1} + (i_{T_1} + i^*_{T_1})s_{r-2}). \tag{2.14}$$

Proof: The proof is essentially the same as in [BerB, Theorem 6.1]. The reader can easily prove it by proceeding as in [BerB, Theorem 6.1]. ∎

In the sequence, we will identify E_r as a subspace of E, $E = E_0 \supset E_1 \supset \cdots \supset E_r \supset \cdots$. Let p_r be the orthogonal projection from E on E_r. Set $p_r^\perp = 1 - p_r$.

For $T > 0$, set

$$g_T^{TZ} = \pi^* g^{TY} \oplus \frac{1}{T^2} g^{TX}. \tag{2.15}$$

Let D_T^Z be the operator defined in (1.10) with respect to g_T^{TZ}, h^F. Set

$$A_T^{(0)} = \frac{1}{2}(d_T^Z + d_T^{Z*}). \tag{2.16}$$

Then by (1.5),

$$A_T^{(0)} = T^{N_X} D_T^Z T^{-N_X}, \quad A_T^{(0)} = \frac{1}{2}(TD^X + D^H + \frac{1}{T}(i_{T_1} + i^*_{T_1})). \tag{2.17}$$

By proceeding as in [BerB, Theorem 6.5], we get

THEOREM 2.2: *For any* $r \geq 2$, $\lambda \in \mathbf{C}$, $\mathrm{Im}(\lambda) \neq 0$, *for any* $s \in E_0$, *when* $T \to +\infty$,

$$(\lambda - T^{r-1} A_T^{(0)})^{-1} s \to p_r(\lambda - \frac{1}{2} D_r)^{-1} p_r s. \tag{2.18}$$

As in [BerB, §6(d)], it follows from Theorem 2.2 that for $r \geq 2$, the eigenvalues of $A_T^{(0)}$ which are $O(1/T^{r-1})$ can be put in one to one correspondence with the corresponding eigenvalues of $\frac{1}{2} D_r$.

3. Functoriality of the analytic torsion form

In this Section, we state our main result.

This Section is organized as follows. In Section 3.1, we define some torsion forms associated to a complex of flat vector bundles. In Section 3.2, we announce our principal Theorem.

We use the notation of Sections 1 and 2.

3.1. TORSION FORM OF A FLAT COMPLEX. Let W be a $\mathcal{C}^\infty$ manifold. Let

$$(E, v)\colon 0 \to E^0 \xrightarrow{v} E^1 \xrightarrow{v} \cdots \xrightarrow{v} E^n \to 0 \tag{3.1}$$

be a flat complex of complex vector bundles on W. That is, $\nabla^E = \bigoplus_{i=0}^n \nabla^{E^i}$ is a flat connection on $E = \bigoplus_{i=0}^n E^i$ and v is a flat chain map, where

$$(\nabla^E)^2 = 0, \quad v^2 = 0, \quad \nabla^E v = 0. \tag{3.2}$$

Put

$$A' = v + \nabla^E. \tag{3.3}$$

Then A' is a flat superconnnection of total degree 1. By [BLo, §2(a)], the cohomology $H(E,v)$ of the complex is a vector bundle on W, and let $\nabla^{H(E,v)}$ be the flat connection on $H(E,v)$ induced by ∇^E.

Let $F^i = v(E^{i-1}), G^i = \mathrm{Ker}(v_{|E^i})$. Then F^i, G^i are flat complex vector bundles on W. We have the following exact sequence of flat vector bundles on W:

$$0 \to G^i \to E^i \xrightarrow{v} F^{i+1} \to 0, \quad 0 \to F^i \to G^i \to H^i(E,v) \to 0. \tag{3.4}$$

Let $h^E = \bigoplus h^{E^i}$, h^H be Hermitian metrics on $E = \bigoplus E^i$, $H(E,v)$. Let h^{F^i}, h^{G^i} be the metrics on F^i, G^i induced by h^{E^i}. Set

$$\begin{aligned} f(\nabla^E, h^E) &= \sum_{i=0}^n (-1)^i f(\nabla^{E^i}, h^{E^i}), \\ f(\nabla^{H(E,v)}, h^H) &= \sum_{i=0}^n (-1)^i f(\nabla^{H^i(E,v)}, h^H). \end{aligned} \tag{3.5}$$

Let $T(A', h^{E^i})$ (resp. $T(A', h^{G^i})$, $T(A', h^E)$) be the torsion form defined in [BLo, Definition 2.20] associated to the first line of (3.4) (resp. the second line of (3.4), (3.1)).

We will say that the flat Hermitian complex (E, A', h^E, h^H) splits if there exist flat Hermitian vector bundles $(F^i, \nabla^{F^i}, h^{F^i})$ such that (E,v) is the complex

$$\begin{aligned} 0 \to F^0 \oplus H^0 \xrightarrow{\mathrm{Id}_{F^0}} F^0 \oplus F^1 \oplus H^1(E,v) \xrightarrow{\mathrm{Id}_{F^1}} F^1 \oplus F^2 \oplus H^2(E,v) \to \cdots \\ \to F^{n-2} \oplus F^{n-1} \oplus H^{n-1}(E,v) \xrightarrow{\mathrm{Id}_{F^{n-1}}} F^{n-1} \oplus H^n(E,v) \to 0. \end{aligned}$$

And for $0 \le i \le n$, $E^i = F^{i-1} \oplus F^i \oplus H^i(E,v)$ $(F^{-1} = F^n = 0)$ is equipped with the metric $h^{E^i} = h^{F^{i-1}} \oplus h^{F^i} \oplus h^{H^i}$.

LEMMA 3.1: *Let $T'(A', h^E, h^H)$ be a real even form on W, verifying the following conditions:*

(a) *The following identity holds,*

$$dT'(A', h^E, h^H) = f(\nabla^E, h^E) - f(\nabla^{H(E,v)}, h^H). \tag{3.6}$$

(b) *If W' is a smooth manifold and $\alpha\colon W' \to W$ is a smooth map, then*

$$T'(\alpha^* A', \alpha^* h^E, \alpha^* h^H) = \alpha^* T'(A', h^E, h^H). \tag{3.7}$$

(c) *If (E, A', h^E, h^H) splits, then $T'(A', h^E) = 0$.*

(d) *The form $T'(A', h^E, h^H)$ depends smoothly on A' and h^E.*

Then

$$T'(A', h^E, h^H) = \sum_{i=0}^{n} (-1)^i \Big(T(A', h^{E^i}) + T(A', h^{G^i}) \Big) \quad \text{in } Q^W / Q^{W,0}. \tag{3.8}$$

Proof: By proceeding as in the proof of [BLo, Theorem A1.2], we get (3.8). ∎

By Lemma 3.1, we get

$$T(A', h^E) = \sum_{i=0}^{n} (-1)^i \Big(T(A', h^{E^i}) + T(A', h^{G^i}) \Big) \quad \text{in } Q^W / Q^{W,0}. \tag{3.9}$$

Let (E, ∇^E) be a flat complex vector bundle on W. Let $0 \subset E^0 \subset \cdots \subset E^n = E$ be a filtration of E such that $\nabla^E_{\cdot}(E^i) \subset E^i$. Let $\mathrm{Gr}^i E = E^i / E^{i-1}$. Then we have a flat complex of complex vector bundles:

$$F^i\colon 0 \to E^i \xrightarrow{v} E^{i+1} \xrightarrow{v} \mathrm{Gr}^{i+1} E \to 0. \tag{3.10}$$

Let h^E, $h^{\mathrm{Gr} E} = \bigoplus_i h^{\mathrm{Gr}^i E}$ be Hermitian metrics on $E, \mathrm{Gr} E = \bigoplus_i \mathrm{Gr}^i E$. Let h^{E^i} be the metric on E^i induced by h^E. Let $h^{F^i} = h^{E^{i-1}} \oplus h^{E^i} \oplus h^{\mathrm{Gr}^i E}$ be the metric on $F^i = E^{i-1} \oplus E^i \oplus \mathrm{Gr}^i E$. Let $T(v + \nabla^{F^i}, h^{F^i})$ be the form on W defined by [BLo, Definition 2.20] associated to (3.10).

Definition 3.1: The torsion form of the filtered flat complex vector bundle E is defined by

$$T(E, \mathrm{Gr}\, E, h^E, h^{\mathrm{Gr} E}) = \sum_{i=0}^{n-1} T(v + \nabla^{F^i}, h^{F^i}). \tag{3.11}$$

3.2. Functoriality of analytic torsion form. Let W, V, S be smooth manifolds. Let $\pi_1\colon W \to V$, $\pi_2\colon V \to S$ be smooth fibrations of manifolds with compact fibre X, Y. Then $\pi_3 = \pi_2 \circ \pi_1\colon W \to S$ is a smooth fibration with compact fiber Z with $\dim Z = n$. Let (F, ∇^F) be a flat complex vector bundle over W. Then we have the diagram of smooth fibrations:

$$\begin{tikzcd}
X \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"'] & W \arrow[d, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
 & Y \arrow[r] & V \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

Let $H^\bullet(X,F_{|X})=\bigoplus_{i=0}^{\dim X} H^i(X,F_{|X})$, $H^\bullet(Z,F_{|Z})$, $H^\bullet(Y,H^\bullet(X,F_{|X}))$ be the $\mathbf{Z}$-graded vector bundles over V, S,S whose fibers over $a\in V$, $s\in S$ are the cohomologies $H^\bullet(X_a,F_{|X_a})$, $H^\bullet(Z_s,F_{|Z_s})$, $H^\bullet(Y_s,H^\bullet(X,F_{|X}))$ of the sheaf of locally flat sections of F, F, $H(X,F_{|X})$ on X_a, Z_s, Y_s.

Let T_1^HW,T_2^HV,T_3^HW be sub-bundles of TW,TV,TW with respect to π_1,π_2,π_3 as in (1.1). Let E be the smooth infinite-dimensional $\mathbf{Z}$-graded vector bundle over S whose fiber over $s\in S$ is $\mathcal{C}^\infty(Z_s,(\Lambda(T^*Z)\otimes F)_{|Z_s})$. For $s\in S$, let $(E_{r,s},d_{r,s})$ be the Leray spectral sequence with respect to $\pi_1\colon Z_s\to Y_s,F$.

PROPOSITION 3.1: *There are flat complex vector bundles $E_r^{p,q}$ ($r\geq 2,p,q\in\mathbf{N}$), and $d_r\colon E_r^{p,q}\to E_r^{p-r,q+1-r}$ such that the fiber of complex $(E_r=\bigoplus_{p,q}E_r^{p,q},d_r)$ over $s\in S$ is the Leray spectral sequence $(E_{r,s}=\bigoplus_{p,q}E_{r,s}^{p,q},d_r)$.*

Proof: By Proposition 1.1, d^W is a superconnection on E, and $d^WF^pE\subset F^pE$; here F^pE is the filtration in Section 2.1.

At first, $E_2^{p,q}=H^p(Y,H^q(X,F_{|X}))(p,q\geq 0)$ are flat vector bundles on S. If $E_r^{p,q}$ ($p,q\in\mathbf{N},r\geq 2$) are flat vector bundles, then d^W induces a flat superconnection on E_r, and E_{r+1} is the cohomology of (E_r,d_r). By [BLo, §2(a)], $E_{r+1}^{p,q}$ are flat vector bundles on S. Now by recurrence, the proof of our Proposition is completed. ∎

By [BLo, §2(a)], there is also a canonical connection $\nabla^{E_r}=\bigoplus_{p,q}\nabla^{E_r^{p,q}}$ on $E_r=\bigoplus_{p,q}E_r^{p,q}$ induced by d^W.

Let g^{TZ},g^{TX},g^{TY} be metrics on TZ,TX,TY. Let h^F be a Hermitian metric on F.

Let $h^{H(X,F_{|X})}$, $h^{H(Z,F_{|Z})}$, $h^{H(Y,H(X,F_{|X}))}$ be the L^2-metrics on $H^\bullet(X,F_{|X})$, $H^\bullet(Z,F_{|Z})$, $H^\bullet(Y,H^\bullet(X,F_{|X}))$ with respect to g^{TX},h^F; g^{TZ},h^F and $g^{TY},h^{H(X,F_{|X})}$ defined in Section 1.2.

Let $\nabla^{TX},\nabla^{TY},\nabla^{TZ}$ be the connections on (TX,g^{TX}), (TY,g^{TY}), (TZ,g^{TZ}) defined in Section 1.1. Let $T^HZ=T_1^HW\cap TZ$. Let $\pi_1^*\nabla^{TY}$ be the connection on $T^HZ\simeq\pi_1^*TY$ induced by ∇^{TY}. Then ${}^0\nabla^{TZ}=\pi_1^*\nabla^{TY}\oplus\nabla^{TX}$ is a connection on $TZ=T^HZ\oplus TX$ which preserves the metric $\pi_1^*g^{TY}\oplus g^{TX}$. Let $\widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})$ be the Chern–Simons $n-1$ forms on Z with values in $o(TZ)$ such that

$$d\widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})=e(TZ,{}^0\nabla^{TZ})-e(TZ,\nabla^{TZ}). \tag{3.12}$$

Let $\mathcal{T}(T_1^HW,g^{TX},h^F),\mathcal{T}(T_2^HV,g^{TY},h^{H(X,F_{|X})})$, $\mathcal{T}(T_3^HW,g^{TZ},h^F)$ be the analytic torsion forms corresponding to π_1,π_2,π_3. Let h^{E_2} be the metric on E_2

induced by $h^{H(Y,H(X,F_{|X}))}$. Let h^{E_r} $(r \geq 3)$ be the L^2 metric on E_r as in Section 2.2. Set

$$\begin{aligned}(3.13)\quad & T(H(Z,F_{|Z}),E_\infty,h^{H(Z,F_{|Z})},h^{E_\infty}) \\ &= \sum_{k=0}^{\dim Z}(-1)^k T(H^k(Z,F_{|Z}),\oplus_{p+q=k}E_\infty^{p,q},h^{H(Z,F_{|Z})},h^{E_\infty}).\end{aligned}$$

Recall that for $r \geq 2$, d_r is induced by d^Z. By (1.6), $d_r + \nabla^{E_r}$ is a flat superconnection of total degree 1 on E_r.

Definition 3.2: Set

$$\begin{aligned}(3.14)\quad T(E_2,H(Z,F_{|Z}),h^{E_2},h^{H(Z,F_{|Z})}) =& \sum_{r=2}^{\infty} T(d_r+\nabla^{E_r},h^{E_r},h^{E_{r+1}}) \\ &- T(H(Z,F_{|Z}),E_\infty,h^{H(Z,F_{|Z})},h^{E_\infty}).\end{aligned}$$

In fact, by [BLo, Theorem 2.24], $T(.,.) \in Q^S/Q^{S,0}$ doesn't depend on the choice of h^{E_r} $(r \geq 3)$ on E_r.

The purpose of this paper is to establish the following result,

THEOREM 3.1: *The following identity holds in $Q^S/Q^{S,0}$,*

$$\begin{aligned}(3.15)\quad \mathcal{T}(T_3^HW,g^{TZ},h^F) =& \int_Y e(TY,\nabla^{TY})\mathcal{T}(T_1^HW,g^{TX},h^F) \\ &+\mathcal{T}(T_2^HV,g^{TY},h^{H(X,F_{|X})}) + T(E_2,H(Z,F_{|Z}),h^{E_2},h^{H(Z,F_{|Z})}) \\ &- \int_Z \widetilde{e}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})f(\nabla^F,h^F).\end{aligned}$$

Remark 3.1: By [BLo, Theorem 3.24], to prove Theorem 3.1, we only need to prove it for a particular choice of T_1^HW, T_2^HV, T_3^HW, and g^{TZ}, g^{TX}, g^{TY}. So we may, and we will, suppose that

$$(3.16)\qquad T_3^HW \subset T_1^HW, \quad g^{TZ} = g^{TX} \oplus \pi_1^* g^{TY}.$$

4. A proof of Theorem 3.1

In this Section, we prove our main result, stated as Theorem 3.1, when g^{TZ}, T_3^HW are given by (3.16).

This Section is organized as follows. In Section 4.1, we introduce a 1-form on $\mathbf{R}_+^* \times \mathbf{R}_+^*$. In Section 4.2, we state seven intermediate results which we need for the proof of Theorem 3.1, whose proofs are delayed to Sections 5–9. In Section 4.3, we prove Theorem 3.1.

Here, we use the assumptions and notation of Sections 1 and 3.2. Recall that f, g are the functions defined in (1.26). Recall also $[a, b]$ is the supercommutator of a, b.

4.1. A FUNDAMENTAL 1-FORM. Recall that $T^H Z = T_1^H W \cap TZ$. Then we have the identification of smooth vector bundles over W,

$$TZ \simeq TX \oplus T^H Z, \quad T^H Z \simeq \pi_1^* TY. \tag{4.1}$$

This identification determines an identification of $\mathbf{Z}$-graded bundles of algebra

$$\Lambda(T^*Z) = \Lambda(T^*Y) \widehat{\otimes} \Lambda(T^*X). \tag{4.2}$$

Let N_X, N_Y, N_Z be the number operators on $\Lambda(T^*X)$, $\Lambda(T^*Y)$, $\Lambda(T^*Z)$. Then N_X, N_Y act naturally on $\Lambda(T^*Z)$. Of course, $N_Z = N_X + N_Y$.

Definition 4.1: For $T \geq 1$, set

$$g_T^{TZ} = \frac{1}{T^2} g^{TX} \oplus \pi_1^* g^{TY}. \tag{4.3}$$

Let $h_T^E = \langle , \rangle_T$ be the scalar product on $E = \Omega(Z, F)$ associated to g_T^{TZ}, h^F defined as in (1.9). Let $C'_{3,u,T}, C''_{3,u,T}, C_{3,u,T}, D_{3,u,T}, \nabla^E, d_T^{Z*}, (i_{T_3})_T^*, *_T$ be the operators defined in Section 1.1 with respect to $(\pi_3, \langle , \rangle_T)$. Let T_1, T_2, T_3 be the tensors defined in (1.4) with respect to $(\pi_1, T_1^H W)$, $(\pi_2, T_2^H V)$, $(\pi_3, T_3^H W)$.

Definition 4.2: Let $\alpha_{u,T}$ be the 1-form with values in Q^S on $\mathbf{R}_+^* \times \mathbf{R}_+^*$,

$$\alpha_{u,T} = \frac{du}{u} \varphi \operatorname{Tr}_s \Big[N_Z f'(D_{3,u^2,T}) \Big] + dT \varphi \operatorname{Tr}_s \Big[\frac{1}{2} *_T^{-1} \frac{\partial *_T}{\partial T} f'(D_{3,u^2,T}) \Big]. \tag{4.4}$$

LEMMA 4.1: *We have*

$$\frac{\partial}{\partial T} C''_{3,u,T} = \Big[C''_{3,u,T}, *_T^{-1} \frac{\partial *_T}{\partial T} \Big], \quad \frac{\partial}{\partial T} C'_{3,u,T} = 0. \tag{4.5}$$

Proof: By Definition, for $s_1, s_2 \in \mathcal{C}^\infty(S, E)$, we have

$$\big\langle \nabla_T^{E*} s_1, s_2 \big\rangle_T = \big\langle s_1, \nabla^E s_2 \big\rangle_T. \tag{4.6}$$

Now, we differentiate (4.6) in the variable T; we get

$$\Big\langle \frac{\partial}{\partial T} \nabla_T^{E*} s_1, s_2 \Big\rangle_T + \Big\langle *_T^{-1} \frac{\partial *_T}{\partial T} \nabla_T^{E*} s_1, s_2 \Big\rangle_T = \Big\langle *_T^{-1} \frac{\partial *_T}{\partial T} s_1, \nabla^E s_2 \Big\rangle_T. \tag{4.7}$$

So we obtain

$$\frac{\partial}{\partial T} \nabla_T^{E*} = \Big[\nabla_T^{E*}, *_T^{-1} \frac{\partial *_T}{\partial T} \Big]. \tag{4.8}$$

In the same way, we have

$$\frac{\partial}{\partial T} d_T^{Z*} = \Big[d_T^{Z*}, *_T^{-1}\frac{\partial *_T}{\partial T}\Big], \quad \frac{\partial}{\partial T}(i_{T_3})_T^* = \Big[(i_{T_3})_T^*, *_T^{-1}\frac{\partial *_T}{\partial T}\Big]. \tag{4.9}$$

By (1.11), (4.8) and (4.9), we get (4.5). ∎

THEOREM 4.1: *We have the following identity,*

$$\begin{aligned} d_{u,T}\alpha_{u,T} = -\frac{1}{u} du dT \varphi \frac{\partial}{\partial b}\Big\{ & d\,\mathrm{Tr}_s\Big[N_Z g(-D^2_{3,u^2,T} + b[C''_{3,u^2,T}, *_T^{-1}\frac{\partial *_T}{\partial T}])\Big] \\ & + d\,\mathrm{Tr}_s\Big[[C'_{3,u^2,T}, N_Z] g(-D^2_{3,u^2,T} + b *_T^{-1}\frac{\partial *_T}{\partial T})\Big]\Big\}_{b=0}. \end{aligned} \tag{4.10}$$

Proof: By (1.11) and (4.5), we know that

$$\Big[D_{3,u^2,T}, \frac{\partial}{\partial T} D_{3,u^2,T}\Big] = -\Big[C'_{3,u^2,T}, \frac{\partial}{\partial T} C''_{3,u^2,T}\Big].$$

By using (4.5), we get

$$\begin{aligned} &\frac{\partial}{\partial T}\mathrm{Tr}_s\Big[N_Z f'(D_{3,u^2,T})\Big] \\ &= \frac{\partial}{\partial b}\mathrm{Tr}_s\Big[N_Z g(-D^2_{3,u^2,T} - b[D_{3,u^2,T}, \frac{\partial}{\partial T} D_{3,u^2,T}])\Big]_{b=0} \\ &= \frac{\partial}{\partial b}\mathrm{Tr}_s\Big[N_Z g(-D^2_{3,u^2,T} + b[C'_{3,u^2,T}, \frac{\partial}{\partial T} C''_{3,u^2,T}])\Big]_{b=0} \\ &= -\frac{\partial}{\partial b}\mathrm{Tr}_s\Big[[C'_{3,u^2,T}, N_Z] g(-D^2_{3,u^2,T} + b\frac{\partial}{\partial T} C''_{3,u^2,T})\Big]_{b=0} \\ &\quad + \frac{\partial}{\partial b}\mathrm{Tr}_s\Big[[C'_{3,u^2,T}, N_Z g(-D^2_{3,u^2,T} + b\frac{\partial}{\partial T} C''_{3,u^2,T})]\Big]_{b=0} \\ &= -\frac{\partial}{\partial b}\mathrm{Tr}_s\Big[[C''_{3,u^2,T}, [C'_{3,u^2,T}, N_Z]] g(-D^2_{3,u^2,T} + b *_T^{-1}\frac{\partial *_T}{\partial T})\Big]_{b=0} \\ &\quad + \frac{\partial}{\partial b}\mathrm{Tr}_s\Big[[C''_{3,u^2,T}, [C'_{3,u^2,T}, N_Z] g(-D^2_{3,u^2,T} + b *_T^{-1}\frac{\partial *_T}{\partial T})]\Big]_{b=0} \\ &\quad + \frac{\partial}{\partial b}\mathrm{Tr}_s\Big[[C'_{3,u^2,T}, N_Z g(-D^2_{3,u^2,T} + b\frac{\partial}{\partial T} C''_{3,u^2,T})]\Big]_{b=0}. \end{aligned} \tag{4.11}$$

Moreover, by (1.5) and (1.11), we know that

$$[C'_{3,u^2,T}, N_Z] = -u\frac{\partial}{\partial u} C'_{3,u^2,T}, \quad [C''_{3,u^2,T}, N_Z] = u\frac{\partial}{\partial u} C''_{3,u^2,T}.$$

From (1.11) and the above equation, we get

$$\begin{aligned} &[C'_{3,u^2,T}, [C''_{3,u^2,T}, N_Z]] - [C''_{3,u^2,T}, [C'_{3,u^2,T}, N_Z]] = -u\frac{\partial}{\partial u} D^2_{3,u^2,T}, \\ &[C'_{3,u^2,T}, [C''_{3,u^2,T}, N_Z]] + [C''_{3,u^2,T}, [C'_{3,u^2,T}, N_Z]] = -[D^2_{3,u^2,T}, N_Z]. \end{aligned} \tag{4.12}$$

By (4.3), we know $[N_Z, *_T^{-1}\frac{\partial *_T}{\partial T}] = 0$. By (1.26) and (4.12),

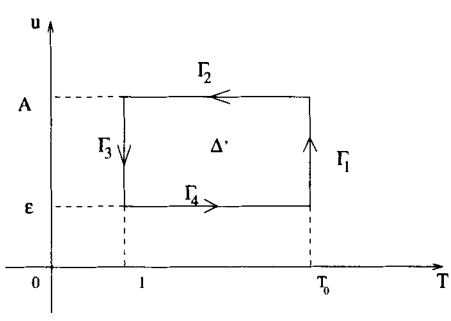
$$\begin{aligned}
&-\frac{\partial}{\partial b}\operatorname{Tr}_s\Big[[C''_{3,u^2,T},[C'_{3,u^2,T},N_Z]]g(-D^2_{3,u^2,T}+b *_T^{-1}\frac{\partial *_T}{\partial T})\Big]_{b=0}\\
&=\frac{1}{2}\frac{\partial}{\partial b}\operatorname{Tr}_s\Big[[D^2_{3,u^2,T},N_Z]g(-D^2_{3,u^2,T}+b *_T^{-1}\frac{\partial *_T}{\partial T})\Big]_{b=0}\\
&\quad-\frac{1}{2}u\frac{\partial}{\partial b}\operatorname{Tr}_s\Big[(\frac{\partial}{\partial u}D^2_{3,u^2,T})g(-D^2_{3,u^2,T}+b *_T^{-1}\frac{\partial *_T}{\partial T})\Big]_{b=0}\\
&=-\frac{1}{2}\operatorname{Tr}_s\Big[*_T^{-1}\frac{\partial *_T}{\partial T}[N_Z,g(-D^2_{3,u^2,T})]\Big]\\
&\quad+\frac{1}{2}u\operatorname{Tr}_s\Big[*_T^{-1}\frac{\partial *_T}{\partial T}\frac{\partial}{\partial u}g(-D^2_{3,u^2,T})\Big]\\
&=-\frac{1}{2}\operatorname{Tr}_s\Big[[N_Z,*_T^{-1}\frac{\partial *_T}{\partial T}]g(-D^2_{3,u^2,T})\Big]+\frac{1}{2}u\frac{\partial}{\partial u}\operatorname{Tr}_s\Big[*_T^{-1}\frac{\partial *_T}{\partial T}g(-D^2_{3,u^2,T})\Big]\\
&=\frac{1}{2}u\frac{\partial}{\partial u}\operatorname{Tr}_s\Big[*_T^{-1}\frac{\partial *_T}{\partial T}g(-D^2_{3,u^2,T})\Big].
\end{aligned}\tag{4.13}$$

By (1.11) and $D_{3,u^2,T}$ is an element of $\Omega(S,\operatorname{End}(E))$, we know that

$$\begin{aligned}
&\operatorname{Tr}_s\Big[[C''_{3,u^2,T},[C'_{3,u^2,T},N_Z]g(-D^2_{3,u^2,T}+b *_T^{-1}\frac{\partial *_T}{\partial T})]\Big]\\
&\qquad=d\operatorname{Tr}_s\Big[[C'_{3,u^2,T},N_Z]g(-D^2_{3,u^2,T}+b *_T^{-1}\frac{\partial *_T}{\partial T})\Big],\\
&\operatorname{Tr}_s\Big[[C'_{3,u^2,T},N_Zg(-D^2_{3,u^2,T}+b\frac{\partial}{\partial T}C''_{3,u^2,T})]\Big]\\
&\qquad=d\operatorname{Tr}_s\Big[N_Zg(-D^2_{3,u^2,T}+b[C''_{3,u^2,T},*_T^{-1}\frac{\partial *_T}{\partial T}])\Big].
\end{aligned}$$

By (4.11) and (4.13), we get Theorem 4.1. ∎

Take $\epsilon, A, T, 0<\epsilon\le 1\le A<+\infty$, $1\le T_0<+\infty$. Let $\Gamma=\Gamma_{\epsilon,A,T_0}$ be the oriented contour in $\mathbf{R}_+^*\times\mathbf{R}_+^*$



The contour Γ consists of four oriented pieces $\Gamma_1, \dots, \Gamma_4$ indicated above. Also Γ bounds an oriented rectangular domain Δ'. For $1 \leq k \leq 4$, set

$$I_k^0 = \int_{\Gamma_k} \alpha. \tag{4.14}$$

Put

(4.15)
$$\theta^0 = -(2\pi i)^{-1/2} \int_{\Delta'} \frac{1}{u} \varphi \frac{\partial}{\partial b} \Big\{ \operatorname{Tr}_s \Big[N_Z g(-D^2_{3,u^2,T} + b[C''_{3,u^2,T}, *_T^{-1} \frac{\partial *_T}{\partial T}]) \Big] + \operatorname{Tr}_s \Big[[C'_{3,u^2,T}, N_Z] g(-D^2_{3,u^2,T} + b *_T^{-1} \frac{\partial *_T}{\partial T}) \Big] \Big\}_{b=0} du dT.$$

THEOREM 4.2: *The following identity holds,*

$$\sum_{k=1}^{4} I_k^0 = d\theta^0. \tag{4.16}$$

Proof: This follows from Theorem 4.1. ∎

4.2. SEVEN INTERMEDIATE RESULTS. Now, we state without proof seven intermediate results, which will play an essential role in the proof of Theorem 3.1. The proofs of these results are delayed to Sections 5–9.

In the sequence, we will assume for simplicity that S is compact. If S is non-compact, the various constants $C > 0$ depend explicitly on the compact set $K \subset S$ on which the given estimate is valid.

Let $Z = \bigcup_{i=1}^k Z_i$ be the decomposition of the connected components of Z. Let $Y_i = \pi_1(Z_i)$. Let $\chi(X), \chi(Y_i), \chi(Z_i), \chi(Z)$ be Euler numbers of the fibers X, Y_i, Z_i, Z. Then $\chi(X)$ is locally constant function on Y. We have

$$\chi(Z_i) = \chi(X)\chi(Y_i). \tag{4.17}$$

In the following, we will also write $\chi(Z) = \sum_i \chi(X)\chi(Y_i)$ as $\chi(X)\chi(Y)$, etc.

Let e_i, f_l be orthonormal bases of $(TX, g^{TX}), (TY, g^{TY})$. Then $\{e_{T,a}\} = \{Te_i, f_l\}$ is the orthonormal base of (TZ, g_T^{TZ}). By [BZ, Proposition 4.15], (1.14),

(4.18)
$$\begin{aligned} *_T^{-1} \frac{\partial *_T}{\partial T} &= -\frac{1}{2} \Big\langle (g_T^{TZ})^{-1} (\frac{\partial}{\partial T} g_T^{TZ}) e_{T,a}, e_{T,b} \Big\rangle_{g_T^{TZ}} c_T(e_{T,a}) \widehat{c}_T(e_{T,b}) \\ &= \frac{1}{T} \sum_i c_T(Te_i) \widehat{c}_T(Te_i) = \frac{1}{T}(2N_X - \dim X). \end{aligned}$$

Let $C_{1,u}, D_{1,u}, C_{2,u}, D_{2,u}$ be the operators defined in (1.11) with respect to $(\pi_1, T_1^H W, g^{TX})$, $(\pi_2, T_2^H V, g^{TY})$. Let $h_T^{H(Z,F_{|Z})}$ be the L^2-metric on $H(Z, F_{|Z})$ with respect to g_T^{TZ}, h^F defined in Section 1.2.

THEOREM 4.3: (i) *For any* $u > 0$,

$$\lim_{T\to+\infty} \varphi \operatorname{Tr}_s \Big[N_Z f'(D_{3,u,T})\Big] = \varphi \operatorname{Tr}_s \Big[N_Z f'(D_{2,u})\Big]. \tag{4.19}$$

(ii) *For any* $u > 0$, *there exist* $C, \delta > 0$ *such that for* $T \geq 1$,

$$\Big|\varphi \operatorname{Tr}_s \Big[*_T^{-1}\frac{\partial *_T}{\partial T} f'(D_{3,u,T})\Big] - \frac{1}{T}\varphi \operatorname{Tr}_s \Big[(2N_X - \dim X) f'(D_{2,u})\Big]\Big| \leq \frac{C}{T^{\delta+1}}. \tag{4.20}$$

(iii) *For* $0 < u_1 < u_2 < +\infty$ *fixed, there exists* $C > 0$ *such that for* $u \in [u_1, u_2]$, $T \geq 1$,

$$\Big|\varphi \operatorname{Tr}_s \Big[N_Z f'(D_{3,u,T})\Big]\Big| \leq C. \tag{4.21}$$

Set

$$\Gamma'_f = \int_1^{+\infty} f'\Big(\frac{i\sqrt{t}}{2}\Big)\frac{dt}{t} + \int_0^1 \Big[f'\Big(\frac{i\sqrt{t}}{2}\Big) - 1\Big]\frac{dt}{t}. \tag{4.22}$$

THEOREM 4.4: *We have the following identity,*

$$\begin{aligned}\lim_{T\to+\infty} \Big\{ &\int_1^{+\infty} \Big\{\frac{1}{2}\varphi \operatorname{Tr}_s[N_Z f'(D_{3,u,T})] - \frac{1}{2}\chi'(Z,F)\Big\}\frac{du}{u} \\ &- \sum_{r\geq 2}(r-1)\Big[\operatorname{Tr}_s[N_{Z|E_r}] - \operatorname{Tr}_s[N_{Z|E_{r+1}}]\Big]\log T\Big\} \\ = &\int_1^{+\infty} \frac{1}{2}\Big\{\varphi \operatorname{Tr}_s[N_Z f'(D_{2,u})] - \operatorname{Tr}_s[N_{Z|E_2}]\Big\}\frac{du}{u} \\ &- \sum_{r\geq 2} T(d_r + \nabla^{E_r}, h^{E_r}, h^{E_{r+1}}) + \frac{1}{2}\Gamma'_f\Big\{\operatorname{Tr}_s[N_{Z|E_2}] - \operatorname{Tr}_s[N_{Z|E_\infty}]\Big\}.\end{aligned} \tag{4.23}$$

THEOREM 4.5: *We have the following identity in* $Q^S/Q^{S,0}$,

$$\begin{aligned}\int_1^{+\infty} \frac{1}{2}\Big\{&\varphi \operatorname{Tr}_s \Big[(2N_X - \dim X) f'\Big(\frac{1}{2}\omega(H(Z,F_{|Z}), h_T^{H(Z,F_{|Z})})\Big)\Big] \\ &- \varphi \operatorname{Tr}_s \Big[(2N_X - \dim X) f'\Big(\frac{1}{2}\omega(E_\infty, h^{E_\infty})\Big)\Big]\Big\}\frac{dT}{T} \\ &= - T(H(Z,F_{|Z}), E_\infty, h^{H(Z,F_{|Z})}, h^{E_\infty}).\end{aligned} \tag{4.24}$$

THEOREM 4.6: *For any* $T \geq 1$,

$$\begin{aligned}&\lim_{\varepsilon\to 0} \varphi \operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon}) f'(D_{3,\varepsilon^2,T/\varepsilon})\Big] \\ &= \frac{2}{T}\int_Y e(TY, \nabla^{TY})\varphi \operatorname{Tr}_s[N_X f'(D_{1,T^2})] - \frac{1}{T}\dim X\,\chi(X)\,\chi(Y)\operatorname{rk}(F).\end{aligned} \tag{4.25}$$

Let ∇_T^{TZ} be the connection on (TZ, g_T^{TZ}) defined in Section 1.1, and let R_T^{TZ} be the curvature of ∇_T^{TZ}.

Put $\widehat{W} = W \times \mathbf{R}_+^*$ and $\widehat{S} = S \times \mathbf{R}_+^*$. Define $\widehat{\pi}_3$: $\widehat{W} \to \widehat{S}$ by $\widehat{\pi}_3(x,T) = (\pi_3(x), T)$. Let ρ be the projection $\widehat{W} \to W$ and let ρ' be the projection $\widehat{W} \to \mathbf{R}_+^*$.

Let $\widehat{Z}$ be the fiber of $\widehat{\pi}_3$. Then $T\widehat{Z} = \rho^* TZ$. Let $g^{T\widehat{Z}}$ be the metric on $T\widehat{Z}$ which coincides with g_T^{TZ} over $W \times \{T\}$. Put $T_3^H \widehat{W} = \rho^* T_3^H W \oplus \rho'^* T\mathbf{R}^*$. Let $\nabla^{T\widehat{Z}}$ be the connection on $T\widehat{Z}$ defined in Section 1.1. By [B4, Theorem 1.1], we get

$$\nabla^{T\widehat{Z}} = \rho^* \nabla_T^{TZ} + dT\Big(\frac{\partial}{\partial T} + \frac{1}{2}(g_T^{TZ})^{-1}\frac{\partial}{\partial T} g_T^{TZ}\Big). \tag{4.26}$$

Then $\nabla^{T\widehat{Z}}$ preserves the metric $g^{T\widehat{Z}}$. The curvature $R^{T\widehat{Z}}$ of $\nabla^{T\widehat{Z}}$ is given by

$$R^{T\widehat{Z}} = \rho^* R_T^{TZ} + dT\Big(\frac{\partial}{\partial T}\nabla_T^{TZ} - \frac{1}{2}\Big[\nabla_T^{TZ}, (g_T^{TZ})^{-1}\frac{\partial}{\partial T} g_T^{TZ}\Big]\Big). \tag{4.27}$$

Definition 4.3: Set (cf. [BZ, Definition 4.19])

(4.28)

$$\widetilde{e}'_T(TZ) = \frac{\partial}{\partial b}\operatorname{Pf}\Big[\frac{1}{2\pi}\Big(R_T^{TZ} + b\Big(\frac{\partial}{\partial T}\nabla_T^{TZ} - \frac{1}{2}\Big[\nabla_T^{TZ}, (g_T^{TZ})^{-1}\frac{\partial}{\partial T} g_T^{TZ}\Big]\Big)\Big)\Big]_{b=0}.$$

By a standard argument in Chern–Weil theory, we know that

$$\frac{\partial}{\partial T}\widetilde{e}(TZ, \nabla_1^{TZ}, \nabla_T^{TZ}) = \widetilde{e}'_T(TZ). \tag{4.29}$$

THEOREM 4.7: *The following identities hold,*

$$\begin{aligned} &\widetilde{e}'_T(TZ) = O(1/T^2) \quad \textit{when } T \to +\infty, \\ &\int_1^{+\infty} \widetilde{e}'_T(TZ)dT = \widetilde{e}(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \quad \textit{in } Q^W/Q^{W,0}. \end{aligned} \tag{4.30}$$

THEOREM 4.8: *There exists $C > 0$ such that for $\varepsilon \in]0,1]$, $\varepsilon \leq T \leq 1$,*

$$\Big|\varphi \operatorname{Tr}_s\Big[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon}) f'(D_{3,\varepsilon^2,T/\varepsilon})\Big] - \frac{2}{\varepsilon}\int_Z \widetilde{e}'_{T/\varepsilon}(TZ) f(\nabla^F, h^F)\Big| \leq C. \tag{4.31}$$

THEOREM 4.9: *There exist $\delta \in]0,1], C > 0$ such that for $\varepsilon \in]0,1]$, $T \geq 1$,*

(4.32)

$$\begin{aligned}\Big|\varphi \operatorname{Tr}_s\Big[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon}) f'(D_{3,\varepsilon^2,T/\varepsilon})\Big] - \frac{1}{T}\varphi \operatorname{Tr}_s\Big[(2N_X - \dim X) f'(D_{2,\varepsilon^2})\Big]\Big| \\ \leq \frac{C}{T^{1+\delta}}.\end{aligned}$$

4.3. Proof of Theorem 3.1. At first, we study individually each I_k^0 ($1 \le k \le 4$), by making in succession $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$. By using (1.33) and (5.4), the above seven intermediate results, and proceeding as in [BerB, §4(c)], [Ma1, §4(c)], we get

$$
\begin{aligned}
I_1^3 = &- \mathcal{T}(T_2^H V, g^{TY}, h^{H(X,F_{|X})}) - \sum_{r=2}^{\infty} \mathcal{T}(d_r + \nabla^{E_r}, h^{E_r}, h^{E_{r+1}}) \\
&+ \frac{1}{2}\Gamma'_f\Big\{\chi(Y)\Big[\frac{1}{2}\dim Y\chi(X)\mathrm{rk}(F) + \chi'(X,F)\Big] - \mathrm{Tr}_s[N_{Z|E_\infty}]\Big\}, \\
I_2^3 = &\mathcal{T}(H(Z,F_{|Z}), E_\infty, h^{H(Z,F_{|Z})}, h^{E_\infty}), \\
I_3^3 = &\mathcal{T}(T_3^H W, g^{TZ}, h^F) + \frac{1}{2}\Gamma'_f\Big\{-\frac{1}{2}\dim Z\chi(Z)\,\mathrm{rk}(F) + \chi'(Z,F)\Big\}, \\
I_4^3 = &- \int_Y e(TY, \nabla^{TY})\mathcal{T}(T_1^H W, g^{TX}, h^F) \\
&+ \int_Z \widetilde{e}(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) f(\nabla^F, h^F) \\
&+ \frac{1}{2}\Gamma'_f\Big\{\frac{1}{2}\dim X\chi(X)\chi(Y)\,\mathrm{rk}(F) - \chi(Y)\chi'(X,F)\Big\}.
\end{aligned} \tag{4.33}
$$

Of course, we must study the right term of (4.16). Note that by [R, §22, Theorem 17], if $\alpha_n \in Q^{S,0}$ is a family of smooth exact forms on S which converge uniformly on any compact set $K \subset S$ to a smooth form α, then $\alpha \in Q^{S,0}$

By (4.16), $\sum_{k=1}^4 I_k^0 \in Q^{S,0}$. Now by analysing the diverging terms appear in succession $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$, as in [BerB, §4(d)], [BG, §9.5], we know easily that $\sum_{k=1}^4 I_k^3 \in Q^{S,0}$. By (4.33), we get Theorem 3.1. ∎

5. Proof of Theorems 4.3, 4.4 and 4.7

This Section is organized as follows. In Section 5.1, we calculate the adiabatic limit of some tensors. In Section 5.2, we calculate the asymptotic expansion of the superconnection A_T when $T \to +\infty$. In Section 5.3, we state two intermediate results, from which Theorem 4.3 follows easily. In Section 5.4, we prove Theorem 4.4. The reader who is only interested in the Ray–Singer metric (i.e., formula (0.6)) can skip this part, and only uses Section 2.2 to prove Theorem 4.4. In Section 5.5, we prove Theorem 4.7.

We use the assumptions of Section 3.2, and we use the notation of Sections 1, 3.2 and 4. Recall also that f, g are the functions defined in (1.26).

5.1. Adiabatic limit of some tensors. In the sequence, if $\alpha_T (T \in [1, +\infty])$

is a family of tensors (resp. differential operators), we write that as $T \to +\infty$,

$$\alpha_T = \alpha_\infty + O(1/T^k),$$

if for any compact set $K \subset W$ and any $p \in \mathbf{N}$, there exists $C > 0$ such that for $T \geq 1$, the sup of the norms of the coefficients of $\alpha_T - \alpha_\infty$ and their derivatives of order $\leq p$ is dominated by C/T^k.

We also use the above notation for tensors or differential operators on V, S.

For $U \in TS, V \in TV$, let $U_3^H \in T_3^H W, U_2^H \in T_2^H V$, $V_1^H \in T_1^H W$ be its horizontal lifts so that $\pi_{3*}U_3^H = U$, $\pi_{2*}U_2^H = U$, $\pi_{1*}V_1^H = V$.

Recall that ∇^{TX}, ∇^{TY}, ∇_T^{TZ} are the connections on (TX, g^{TX}), (TY, g^{TY}), (TZ, g_T^{TZ}) defined in Section 1.1. Let T_1, T_2, T_3 be the tensors defined in (1.4) with respect to π_1, π_2, π_3. Let $S_1, S_2, S_{3,T}$ be the tensors defined in Section 1.1 associated to $(\pi_1, T_1^H W, g^{TX})$, $(\pi_2, T_2^H V, g^{TY})$, $(\pi_3, T_3^H W, g_T^{TZ})$. Then for $g_\alpha, g_\beta \in TS$,

$$T_3(g_\alpha, g_\beta) = [T_2(g_\alpha, g_\beta)]_2^H + T_1(g_{\alpha,2}^H, g_{\beta,2}^H). \tag{5.1}$$

By (4.1), we have the identification of vector bundles on W,

$$TW = T_3^H W \oplus T^H Z \oplus TX. \tag{5.2}$$

Let $P^{TX}, P^{T_3^H W}, P^{T^H Z}$, P^{TZ} be the corresponding projections from TW on $TX, T_3^H W$, $T^H Z, TZ$.

Let ${}^0\nabla^{TZ} = \pi_1^* \nabla^{TY} \oplus \nabla^{TX}$ be the connection on $TZ \simeq T^H Z \oplus TX$. Recall that S_1 is a 1-form on W with values in the antisymmetric element of $\mathrm{End}(TW)$, and for $X \in TW$, $S_1(X)$ maps TX to $T_1^H W$ (resp. $T_1^H W$ to TX).

Definition 5.1: Let $A_{3,\infty}$, $A_{3,\infty}^*$ be the 1-forms on W with values in $\mathrm{End}(TZ)$ defined by: for $X \in TW, Y, Z \in TZ$,

$$\begin{aligned} A_{3,\infty}(X)Y &= P^{TX}\Big\{S_1(X)P^{T^H Z}Y\Big\}, \\ \langle A_{3,\infty}^*(X)Y, Z\rangle_{g^{TZ}} &= \langle Y, A_{3,\infty}(X)Z\rangle_{g^{TZ}}. \end{aligned} \tag{5.3}$$

THEOREM 5.1: *The connection $\nabla_\infty^{TZ} = {}^0\nabla^{TZ} + A_{3,\infty}$ preserves TX, and its restriction to TX is equal to ∇^{TX}. We have*

$$\nabla_T^{TZ} = {}^0\nabla^{TZ} + A_{3,\infty} - \frac{1}{T^2}A_{3,\infty}^*. \tag{5.4}$$

Proof: On each fiber Z, ∇_T^{TZ} is the Levi–Civita connection of (TZ, g_T^{TZ}). By [BCh, (4.14), (4.15)], we get (5.4) along the fibres Z.

Now we consider $\pi_1^* g^{TY}, g^{TX}$ as tensors on $TZ = T^H Z \oplus TX$, by extending to 0 on the complements. For $U \in TS$, let $L_{U_3^H}$ be the Lie derivative operator acting on the tensor algebra of TZ. Then for Y (resp. X) $\mathcal{C}^\infty$ section of TY (resp. TX), let $Z = Y_1^H + X$. By using [B4, (1.5), (1.8)], and $[U_3^H, X] \in TX$, we get

$$\tag{5.5} \begin{aligned} (L_{U_3^H} g^{TX})(Y_1^H, X) =& L_{U_3^H}(g^{TX}(Y_1^H, X)) - g^{TX}(L_{U_3^H} Y_1^H, X) \\ & - g^{TX}(Y_1^H, L_{U_3^H} X) \\ =& - \left\langle [U_3^H, Y_1^H], X \right\rangle_{g^{TX}} = -2 \left\langle S_1(U_3^H) Y_1^H, X \right\rangle_{g^{TX}}, \\ [U_3^H, Y_1^H] =& [U_2^H, Y]_1^H + P^{TX}[U_3^H, Y_1^H]. \end{aligned}$$

By [B4, Theorem 1.1], (5.5), we get

$$\tag{5.6} \begin{aligned} \nabla^{TZ}_{T, U_3^H} Z =& L_{U_3^H} Z + \frac{1}{2}(g_T^{TZ})^{-1}(L_{U_3^H} g_T^{TZ})(Z) \\ =& [U_3^H, Z] + \frac{1}{2}(g^{TY})^{-1} P^{T^H Z}(L_{U_3^H} g_T^{TZ})(Z) \\ & + \frac{1}{2}(g^{TX})^{-1} P^{TX}(L_{U_3^H} g^{TX})(Z) \\ =& (\nabla^{TY}_{U_2^H} Y)^H + \nabla^{TX}_{U_3^H} X + A_{3,\infty}(U_3^H) Y_1^H - \frac{1}{T^2} A_{3,\infty}^*(U_3^H) X. \end{aligned}$$

By (5.4), we also get the property of ∇^{TZ}_∞. The proof of Theorem 5.1 is completed. ∎

THEOREM 5.2: (i) *For $X \in TX, Y \in TW, Y' \in T_3^H W$,*

$$\tag{5.7} T^2 \left\langle S_{3,T}(X) Y, Y' \right\rangle_T = \left\langle S_1(X) Y, Y' \right\rangle.$$

(ii) *For $X \in TY, U \in TS, Y \in TV$,*

$$\tag{5.8} \left\langle S_{3,T}(X_1^H) Y_1^H, U_3^H \right\rangle_T = \left\langle S_2(X) Y, U_2^H \right\rangle.$$

Proof: By using (5.4), [B4, (1.5)], and proceeding as in [Ma1, (1.28), (1.30)], we get Theorem 5.2. Comparing to [Ma1, Theorem 1.7], the horizontal space $T_3^H W$ doesn't change here, so the final formula is simpler. ∎

5.2. ASYMPTOTICS OF THE SUPERCONNECTION A_T WHEN $T \to +\infty$. Let dv_X, dv_Y, dv_Z be the Riemannian volume forms on X, Y, Z with respect to g^{TX}, g^{TY}, g^{TZ}. Let $\langle\ \rangle_{\Lambda(T^*Z)\otimes F}$ be the metric on $\Lambda(T^*Z) \otimes F$ induced by g^{TZ}, h^F. Recall that D^X, D^H are the operators defined in (2.8) along the fibres Z.

Definition 5.2: For $a \in V$, $s \in S$, let $E_a, E_{0,s}$ (resp. $E_{1,s}$) be the vector spaces of the $\mathcal{C}^\infty$ sections of $\Lambda(T^*Z) \otimes F$ on X_a, Z_s (resp. $\operatorname{Ker} D^X$ on Y_s).

For $a \in V, s, s' \in E_a$, put

$$\langle s, s'\rangle_{|E_a} = \int_{X_a} \langle s, s'\rangle_{\Lambda(T^*Z)\otimes F}\, dv_X. \tag{5.9}$$

For $\mu \in \mathbf{R}$, $s \in S$, let $E^\mu_{0,s}$, $E^\mu_{1,s}$ be the Sobolev spaces of order μ of sections of $\Lambda(T^*Z) \otimes F$, $\operatorname{Ker} D^X$ on Z_s, Y_s.

For $a \in V$, let p_a be the projection from E_a on $\operatorname{Ker} D^X$ with respect to $\langle\ \ \rangle_{|E_a}$, and let $p^\perp = 1 - p$. Let $\langle\ \ \rangle$ be the Hermitian product on E_0 with respect to the metrics g^{TZ}, h^F as in (1.9). Let $E_1^{0,\perp}$ be the orthogonal bundle to E_1^0 in $(E_0^0, \langle\ \ \rangle)$.

Definition 5.3: For $T \geq 1$, set

$$A_{u,T} = T^{N_X} C_{3,u^2,T} T^{-N_X}, \quad A_T = A_{1,T}. \tag{5.10}$$

Then

$$\begin{aligned} &\varphi \operatorname{Tr}_s \left[N_Z f'(D_{3,u^2,T})\right] = \varphi \operatorname{Tr}_s \left[N_Z g(A^2_{u,T})\right], \\ &\varphi \operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} f'(D_{3,u^2,T})\right] = \varphi \operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} g(A^2_{u,T})\right]. \end{aligned} \tag{5.11}$$

Let $A_T^{(0)}$ (resp. $A_T^{(>0)}$) be the part of A_T of degree 0 (resp. > 0) in $\Lambda(T^*S)$. Let $T_{1|TY}$ be the restriction of T_1 on TY. Let $\nabla_T^{E,u}$ be the Hermitian connection on E_0 defined as in (1.10) with respect to g_T^{TZ}. Let $c_T(\cdot)$, (resp. $c(\cdot)$, $\widehat{c}(\cdot)$) be the Clifford action of TZ on $\Lambda(T^*Z)$ with respect to g_T^{TZ} (resp. g^{TZ}) defined as in (1.14). Then by (1.17) and (2.17),

$$\begin{aligned} A_T^{(0)} &= \frac{1}{2}\Big(TD^X + D^H + \frac{1}{T}(i_{T_{1|TY}} + i^*_{T_{1|TY}})\Big), \\ A_T^{(>0)} &= T^{N_X}\left(\nabla_T^{E,u} - \frac{1}{2} c_T(T_3)\right) T^{-N_X}. \end{aligned} \tag{5.12}$$

Let f_l, e_i be orthonormal bases of $(TY, g^{TY}), (TX, g^{TX})$. Let $\{g_\alpha\}$ be a basis of TS. Let $k_1, k_2, k_{3,T}$ be the horizontal 1-forms on W, V, W associated to (π_1, g^{TX}), (π_2, g^{TY}), (π_3, g_T^{TZ}) defined in (1.13). Let $\nabla_T^{TZ\otimes F,u}$, ${}^0\nabla^{TZ\otimes F}$, ${}^0\nabla^{TZ\otimes F,u}$ be the connections on $\Lambda(T^*Z)\otimes F$ induced by $(\nabla_T^{TZ}, \nabla^{F,u})$, $({}^0\nabla^{TZ}, \nabla^F)$, $({}^0\nabla^{TZ}, \nabla^{F,u})$. For U a vector field on S, $s \in \mathcal{C}^\infty(S, E_0)$, set

$$\begin{aligned} {}^0\nabla^E{}_U s &= {}^0\nabla^{TZ\otimes F}{}_{U_3^H} s, \\ {}^0\nabla^{E,u}{}_U s &= \frac{1}{2} k_1(U_3^H) s + \frac{1}{2} k_2(U_2^H) s + {}^0\nabla^{TZ\otimes F,u}{}_{U_3^H} s. \end{aligned} \tag{5.13}$$

Let $\mathcal{C}$ be the superconnection on E_0,

$$\mathcal{C} = {}^0\nabla^{E,u} + \frac{1}{2}D^H - \frac{1}{2}c(T_2). \tag{5.14}$$

THEOREM 5.3: *When $T \to +\infty$, we have*

$$A_T = \frac{1}{2}TD^X + \mathcal{C} + O(1/T). \tag{5.15}$$

Proof: By Theorem 5.2,

$$k_{3,T}(g_\alpha) = k_1(g_\alpha) + k_2(g_\alpha). \tag{5.16}$$

By Theorem 5.1, we get

$$\begin{aligned} T^{N_X}\nabla_T^{TZ\otimes F,u}T^{-N_X} = & {}^0\nabla^{TZ\otimes F,u} \\ & + \frac{1}{2T}\langle A_{3,\infty}(\cdot)f_l, e_j\rangle_{g^{TX}}\Big(c(f_l)c(e_j) - \widehat{c}(f_l)\widehat{c}(e_j)\Big). \end{aligned} \tag{5.17}$$

By (5.1), we have

$$T^{N_X}c_T(T_3(g_\alpha, g_\beta))T^{-N_X} = c(T_2(g_\alpha, g_\beta)) + \frac{1}{T}c(T_1(g_\alpha, g_\beta)). \tag{5.18}$$

By (1.16), (1.17) and (5.16)–(5.18), we get (5.15). ∎

THEOREM 5.4: *For any $T \in [1, +\infty]$, the operator pA_Tp is a superconnection on E_1. When $T \to +\infty$,*

$$pA_Tp = C_{2,1} + O(1/T). \tag{5.19}$$

Proof: Let $\nabla^{H(X,F_{|X}),u}$ be the connection on $(H(X, F_{|X}), h^{H(X,F_{|X})})$ defined by (1.21). Let $\nabla^{\Omega(X,F_{|X}),u}$ be the connection on $\Omega(X, F_{|X})$ as in (1.16) corresponding to g^{TX}, h^F. Then by Proposition 1.1,

$$\nabla^{H(X,F_{|X}),u} = p\nabla^{\Omega(X,F_{|X}),u}p. \tag{5.20}$$

By using (5.14) and (5.20), we get

$$p\mathcal{C}p = C_{2,1}. \tag{5.21}$$

By Theorem 5.3 and (5.21), we get (5.19). ∎

5.3. THE MATRIX STRUCTURE OF A_T^2. Let $\nabla^{\Lambda(T^*X)}$ be the connection on $\Lambda(T^*X)$ induced by ∇^{TX}. Let $R^{\Lambda(T^*X)}$ be the curvature of $\nabla^{\Lambda(T^*X)}$. Let $\Psi = \omega(F,h^F)$. Let $R^{F,u}$ be the curvature of $\nabla^{F,u}$. By (1.7) and (1.8),

$$R^{F,u} = -\frac{1}{4}\Psi^2. \tag{5.22}$$

Put

$$\begin{aligned} E_T &= pA_T^2p, \quad F_T = pA_T^2p^\perp, \\ G_T &= p^\perp A_T^2 p, \quad H_T = p^\perp A_T^2 p^\perp. \end{aligned} \tag{5.23}$$

Then we write A_T^2 in matrix form with respect to the splitting $E_0^0 = E_1^0 \oplus E_1^{0,\perp}$,

$$A_T^2 = \begin{bmatrix} E_T & F_T \\ G_T & H_T \end{bmatrix}.$$

THEOREM 5.5: *There exist operators E, F, G, H such that, as $T \to +\infty$,*

$$\begin{aligned} E_T &= E + O(1/T), \quad F_T = TF + O(1), \\ G_T &= TG + O(1), \quad H_T = T^2H + O(T). \end{aligned} \tag{5.24}$$

Let

$$Q_\infty = \frac{1}{2}[D^X, \mathcal{C}]. \tag{5.25}$$

Then $Q_\infty(E_1^0) \subset E_1^{0,\perp}$, and Q_∞ is a smooth family of first order elliptic operators acting along the fibres X,

$$\begin{aligned} Q_\infty =& \frac{1}{4}\sum_{i,l} c(e_i)c(f_{l,1}^H)\Big[(R^{\Lambda(T^*X)} + R^{F,u})(e_i, f_{l,1}^H) - {}^0\nabla^{TZ\otimes F,u}_{T_1(e_i,f_{l,1}^H)}\Big] \\ &+ \frac{1}{2}\sum_{i,\alpha} c(e_i)g^\alpha\Big[(R^{\Lambda(T^*X)} + R^{F,u})(e_i, g_{\alpha,3}^H) - {}^0\nabla^{TZ\otimes F,u}_{T_1(e_i,g_{\alpha,3}^H)}\Big] \\ &- \frac{1}{4}\widehat{c}(e_i)\widehat{c}(f_{l,1}^H)R^{F,u}(e_i, f_{l,1}^H) - \frac{1}{8}c(f_{l,1}^H)\widehat{c}(e_i)({}^0\nabla^{TZ\otimes F,u}_{f_{l,1}^H}\Psi)(e_i) \\ &- \frac{1}{8}c(e_i)\widehat{c}(f_{l,1}^H)({}^0\nabla^{TZ\otimes F,u}_{e_i}\Psi)(f_{l,1}^H) - \frac{1}{4}g^\alpha\widehat{c}(e_i)({}^0\nabla^{TZ\otimes F,u}_{g_\alpha}\Psi)(e_i) \\ &+ \frac{1}{4}c(e_i)g^\alpha e_i(k_1(g_\alpha)). \end{aligned} \tag{5.26}$$

Moreover,

$$\begin{aligned} E &= p\mathcal{C}^2p, \qquad F = pQ_\infty p^\perp, \\ G &= p^\perp Q_\infty p, \quad H = \tfrac{1}{4}p^\perp D^{X,2}p^\perp. \end{aligned} \tag{5.27}$$

Proof: By (1.16) and (2.6), we have

$$
\begin{aligned}
D^X &= c(e_i)^0\nabla_{e_i}^{TZ\otimes F,u} - \frac{1}{2}\widehat{c}(e_i)\Psi(e_i),\\
D^H &= c(f_{l,1}^H)^0\nabla_{f_{l,1}^H}^{TZ\otimes F,u} - \frac{1}{2}\widehat{c}(f_{l,1}^H)\Psi(f_{l,1}^H).
\end{aligned} \tag{5.28}
$$

By (5.12), (5.28) and Theorem 5.3, the proof is as same as [Ma1, Theorem 5.10]. ∎

By (5.21) and (5.27), we know

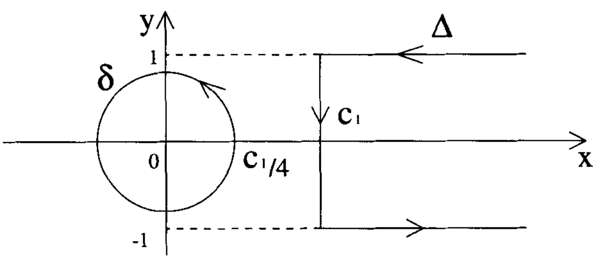
$$
C_{2,1}^2 = p(E - FH^{-1}G)p. \tag{5.1}
$$

5.4. Two intermediate results. If C is an operator, let $\operatorname{Sp}(C)$ be the spectrum of C.

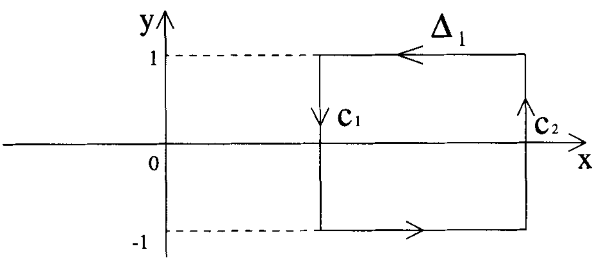
Recall that $D_r = d_r + d_r^*$ is defined in (2.11). For $r \geq 2$, $s \in S$, set $\operatorname{Sp} D_{r,s}^2$ (resp. $\operatorname{Sp} D_{r,s}^{2,>0}$) to be the spectrum (resp. the positive spectrum) of $\operatorname{Sp} D_{r,s}^2$. The constants $c_1, c_2 > 0$ are fixed once and for all such that

$$
\bigcup_{\substack{r\geq 2\\ s\in S}} \operatorname{Sp} D_{r,s}^{2,>0} \subset]4c_1, 4c_2[\quad \text{and} \quad]0, 8c_1[\cap \bigcup_{s\in S} \operatorname{Sp} D_s^{Y,2} = \emptyset. \tag{5.30}
$$

Let $\delta, \Delta \subset \mathbf{C}$ be the contour in $\mathbf{C}$,



Let Δ_1 be the contour in $\mathbf{C}$,



Let δ', Δ'_1 be the domains which are bounded by δ, Δ_1.

By (2.18), for $r \geq 2$, the eigenvalues of $A_T^{(0)}$ which are $O(1/T^{r-1})$ can be put in one to one correspondence with the corresponding eigenvalues of $\frac{1}{2}D_r$. So there exists $T_0 > 0$ such that for $T > T_0$,

$$\operatorname{Sp} A_T^{(0),2} \cap [0, 2c_1[\subset \frac{\delta'}{T^{2(n-1)}} \cup \bigcup_{i=2}^{n} \frac{\Delta'_1}{T^{2(i-1)}}. \tag{5.31}$$

By [B4, Proposition 9.2], for any $u > 0$, $T > 1$,

$$\operatorname{Sp} A_{u,T}^2 = \operatorname{Sp}(u^2 A_T^{(0),2}). \tag{5.32}$$

By (1.26) and (5.32), it is clear that for $u > 0$, $T \geq T_0$,

$$g(u^2 A_T^2) = \frac{1}{2\pi i} \int_{\delta \cup \Delta} \frac{g(u^2 \lambda)}{\lambda - A_T^2} d\lambda. \tag{5.33}$$

Set

$$F_u(A_T^2) = \frac{1}{2\pi i} \int_{\Delta} \frac{g(u^2\lambda)}{\lambda - A_T^2} d\lambda, \quad F_u(C_{2,1}^2) = \frac{1}{2\pi i} \int_{\Delta} \frac{g(u^2\lambda)}{\lambda - C_{2,1}^2} d\lambda. \tag{5.34}$$

Let $P_{u,T}(x,x'), F_u(A_T^2)(x,x')\, (x,x' \in Z_s)$ be the $\mathcal{C}^\infty$ kernels of the operators $g(u^2 A_T^2)$, $F_u(A_T^2)$ calculated with respect to $dv_Z(x')$.

Let ψ_u be the map from $\Lambda(T^*S)$ to $\Lambda(T^*S)$ defined by

$$\alpha \in \Lambda(T^*S) \to u^{-\deg \alpha} \alpha \in \Lambda(T^*S). \tag{5.35}$$

As [Ma1, Proposition 5.14], for $u > 0$, $T > 0$, we have

$$\begin{aligned} &\operatorname{Tr}_s \Big[N_Z \frac{1}{2\pi i} \int_\Delta \frac{g(u^2\lambda)}{\lambda - \frac{1}{u^2}(A_{u,T}^2)} d\lambda \Big] = \psi_u \operatorname{Tr}_s \Big[N_Z F_u(A_T^2) \Big], \\ &\operatorname{Tr}_s \Big[N_Z g(A_{u,T}^2) \Big] = \psi_u \operatorname{Tr}_s \Big[N_Z g(u^2 A_T^2) \Big], \\ &\operatorname{Tr}_s \Big[(2N_X - \dim X) g(A_{u,T}^2) \Big] = \psi_u \operatorname{Tr}_s \Big[(2N_X - \dim X) g(u^2 A_T^2) \Big]. \end{aligned} \tag{5.36}$$

By proceeding as in [Ma1, Theorems 5.19–5.25], we have

THEOREM 5.6: (i) *For $m \in \mathbf{N}$, $0 < u_1 < u_2 < +\infty$ fixed, there exists $C > 0$ such that for $x, x' \in Z_s$, $u \in [u_1, u_2]$, $T \geq T_0$,*

$$\sup_{|\alpha|, |\alpha'| \leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial x^\alpha \partial x'^{\alpha'}} P_{u,T}(x,x') \right| \leq C. \tag{5.37}$$

(ii) *For $m \in \mathbf{N}$, there exist $c > 0, C' > 0$ such that for $x, x' \in Z_s$, $u \geq u_0$, $T \geq T_0$,*

$$\sup_{|\alpha|,|\alpha'|\leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial x^\alpha \partial x'^{\alpha'}} F_u(A_T^2)(x,x') \right| \leq c\exp(-C'u^2). \tag{5.38}$$

For $A \in L(E_0^0, E_0^0)$, we denote by $||A||^{0,0}$ the norm of A with respect to $|\ \ |_0$.

THEOREM 5.7: (i) *For $0 < u_1 < u_2 < +\infty$ fixed, there exists $C > 0$ such that for $u \in [u_1, u_2]$, $T \geq T_0$,*

$$\left\| g(u^2 A_T^2) - p g(u^2 C_{2,1}^2) p \right\|^{0,0} \leq c/T^{1/4}. \tag{5.39}$$

(ii) *There exist $c > 0, C > 0$ such that for $u \geq u_0, T \geq T_0$,*

$$\left\| F_u(A_T^2) - p F_u(C_{2,1}^2) p \right\|^{0,0} \leq c/T^{1/4}. \tag{5.40}$$

5.5. PROOF OF THEOREM 4.3. By (4.18), (5.36), (5.37), (5.39), and by proceeding as in [BL, §11(p) and 13(q)], we get Theorem 4.3.

By (5.38), (5.40), and by proceeding as in [BL, §11(p) and 13(q)], we get the following result which will be used in the proof of Theorem 4.4.

THEOREM 5.8: *There exist $\delta \in]0,1], C > 0$ such that for $u \geq u_0$, $T \geq T_0$,*

$$\left| \mathrm{Tr}_s\left[N_Z F_u(A_T^2)\right] - \mathrm{Tr}_s\left[N_Z F_u(C_{2,1}^2)\right] \right| \leq C/T^\delta. \tag{5.41}$$

5.6. PROOF OF THEOREM 4.4. We use the notation in Sections 2.2 and 3.2.

Recall that $(E_r, d_r)(r \geq 2)$ is the Leray spectral sequence associated to $\pi_1\colon Z_s \to Y_s$ and F. Let $\nabla^{E_r *}(r \geq 2)$ be the adjoint of ∇^{E_r} with respect to h^{E_r}. Let $\nabla^{E_r,u} = \frac{1}{2}(\nabla^{E_r} + \nabla^{E_r *})$. By [BLo, Proposition 2.6], we have

$$\nabla^{E_{r+1}} = p_r \nabla^{E_r} p_r. \tag{5.42}$$

By recurrence and (1.22), (5.13), we get

$$\nabla^{E_r} = p_r {}^0\nabla^E p_r. \tag{5.43}$$

So we have

$$\nabla^{E_r,u} = p_r {}^0\nabla^{E,u} p_r. \tag{5.44}$$

For $u > 0$, $r > 1$, put

$$\mathcal{C}_{r,u} = \nabla^{E_r,u} + \frac{1}{2} u D_r. \tag{5.45}$$

For $r \geq 2$, $T \gg 1$, put

$$
(5.46) \qquad \begin{aligned} \widetilde{p}_{r,T} &= \frac{1}{2\pi i} \int_{\{\lambda \in \mathbf{C}, |\lambda| = \sqrt{c_2}\}} (\lambda - T^{r-1} A_T^{(0)})^{-1} d\lambda, \\ \widetilde{p}^{\perp}_{r,T} &= 1 - \widetilde{p}_{r,T}, \quad q_{r,T} = \widetilde{p}_{r,T} - \widetilde{p}_{r+1,T}. \end{aligned}
$$

Let $\widetilde{p}_{r,T}(x,x')$, $p_r(x,x')$ $(x,x' \in Z_s, s \in S)$ be the $\mathcal{C}^\infty$ kernels of the operators $\widetilde{p}_{r,T}$, p_r with respect to $dv_Z(x')$.

To follow [Ma2, §2], the following result is crucial.

PROPOSITION 5.1: *For any $m \in \mathbf{N}$, there exists $\delta > 0$ such that under the norm $\mathcal{C}^m$, for $r \geq 2$, $T \gg 1$, we have*

$$
(5.47) \qquad \widetilde{p}_{r,T}(x,x') = p_r(x,x') + O(1/T^\delta).
$$

Proof: By proceeding as in [Ma2, Proposition 2.12], we get (5.47). ∎

Let $(T^{k-1}A_T^{(0)}\widetilde{p}_{k,T})(x,x')$, $(D_k p_k)(x,x')$ $(x,x' \in Z_s, s \in S, k \geq 2)$ be the $\mathcal{C}^\infty$ kernels of the operators $T^{k-1}A_T^{(0)}\widetilde{p}_{k,T}$, $D_k p_k$ with respect to $dv_Z(x')$. By (5.46), as in [Ma2, (2.62)], we get

$$
(5.48) \qquad (T^{k-1}A_T^{(0)}\widetilde{p}_{k,T})(x,x') = (D_k p_k)(x,x') + O(1/T^\delta).
$$

For $u > 0$, set

$$
(5.49) \qquad A_{r,u,T} = A_{T^{r-1}u,T}.
$$

For $2 \leq r \leq n = \dim Z$, $T \geq T_0$, set

$$
(5.50) \qquad \begin{aligned} F_{r,u,T} &= \frac{1}{2\pi i} \psi_u \varphi \operatorname{Tr}_s \Big[N_Z \int_{\Delta_1} g(u^2\lambda)(\lambda - A^2_{r,1,T})^{-1} d\lambda \Big], \\ F_{r,u,\infty} &= \frac{1}{2\pi i} \psi_u \varphi \operatorname{Tr}_s \Big[N_Z \int_{\Delta_1} g(u^2\lambda)(\lambda - \mathcal{C}^2_{r,1})^{-1} d\lambda \Big], \\ G_{r,u,T} &= \frac{1}{2\pi i} \psi_u \varphi \operatorname{Tr}_s \Big[N_Z \int_{\delta} g(u^2\lambda)(\lambda - A^2_{r,1,T})^{-1} d\lambda \Big] \quad \text{for } r \geq 1, \\ G_{1,u,\infty} &= \frac{1}{2\pi i} \psi_u \varphi \operatorname{Tr}_s \Big[N_Z \int_{\delta} g(u^2\lambda)(\lambda - C^2_{2,1})^{-1} d\lambda \Big], \\ G_{r,u,\infty} &= \frac{1}{2\pi i} \psi_u \varphi \operatorname{Tr}_s \Big[N_Z \int_{\delta} g(u^2\lambda)(\lambda - \mathcal{C}^2_{r,1})^{-1} d\lambda \Big]. \end{aligned}
$$

Then for $2 \leq r \leq n$,

$$
(5.51) \qquad F_{r,u,\infty} + G_{r,u,\infty} = \varphi \operatorname{Tr}_s \Big[N_Z g(\mathcal{C}^2_{r,u}) \Big].
$$

By using (5.47) and (5.48), and proceeding as in [Ma2, Theorem 2.19], with necessary modification, we get

THEOREM 5.9: (i) *There exist* $\delta > 0$, $C_1 > 0$, $C > 0$, $T_0 > 0$ *such that for* $u \geq 1$, $T \geq T_0$, $2 \leq r \leq n$, *we have*

$$|F_{r,u,T} - F_{r,u,\infty}| \leq \frac{C}{T^\delta} e^{-C_1 u}. \tag{5.52}$$

(ii) *There exist forms* $a_{r,i,T}, b_{n,i,T}$ $(T \in [T_0, +\infty], 1 \leq r \leq n, -\dim S \leq i \leq 0)$, C^∞ *on* S, *such that*

$$\begin{aligned} G_{r,u,\infty} &= -\sum_{i=-\dim S}^{-1} a_{r,i,\infty} u^i + \operatorname{Tr}_s \left[N_{Z|E_{r+1}}\right], \\ G_{n,u,T} &= \sum_{i=-\dim S}^{-1} b_{n,i,T} u^i + \chi'(Z, F). \end{aligned} \tag{5.53}$$

When $u \to 0$, *uniformly for* $T \geq T_0$, *we have*

$$\begin{aligned} F_{r,u,T} &= \sum_{i=-\dim S}^{0} a_{r,i,T} u^i + O(u) \quad \text{for } r \geq 2, \\ G_{1,u,T} &= -\sum_{i=-\dim S}^{0} a_{1,i,T} u^i + O(u). \end{aligned} \tag{5.54}$$

(iii) *We have*

$$a_{r,0,\infty} = \operatorname{Tr}_s \left[N_{Z|E_r}\right] - \operatorname{Tr}_s \left[N_{Z|E_{r+1}}\right] \quad \text{for } r \geq 2. \tag{5.55}$$

There exists $\delta > 0$ *such that for* $1 \leq r \leq n$, *when* $T \to +\infty$, *we have*

$$\begin{aligned} a_{r,i,T} &= a_{r,i,\infty} + O(1/T^\delta), \\ b_{n,i,T} &= -a_{n,i,\infty} + O(1/T^\delta) \quad \text{for } i < 0, \\ b_{n,i,T} T^{-(n-1)i} &+ \sum_{r=2}^{n} a_{r,i,T} T^{-(r-1)i} = -a_{1,i,T} \quad \text{for } i < 0. \end{aligned} \tag{5.56}$$

As in [Ma2, (2.106)], for $T \geq T_0$, we have

$$G_{1,u,T} = \sum_{r=2}^{n} F_{r,T^{-r+1}u,T} + G_{n,T^{-(n-1)}u,T}. \tag{5.57}$$

By using Theorems 5.8, 5.9 and (5.57), and proceeding as in [Ma2, §2(e), (f)], by [BLo, Definition 2.20], we get Theorem 4.4. ∎

5.7. PROOF OF THEOREM 4.7. We use the notation of Section 4.2. Let

$$'\nabla_T^{TZ} = T^{-N_X}\nabla_T^{TZ}T^{N_X}, \quad '\nabla^{T\widehat{Z}} = T^{-N_X}\nabla^{T\widehat{Z}}T^{N_X}. \tag{5.58}$$

Let $'R_T^{TZ}$ be the curvature of $'\nabla_T^{TZ}$. Then $'\nabla^{T\widehat{Z}}$ preserves the metric $\rho^* g^{TZ}$ on $T\widehat{Z}$. By (5.4),

$$'\nabla_T^{TZ} = {}^0\nabla^{TZ} + \frac{1}{T}(A_{3,\infty} - A_{3,\infty}^*). \tag{5.59}$$

By (4.28), we get

(5.60)

$$\widetilde{e}'_T(TZ) = \frac{\partial}{\partial b}\mathrm{Pf}\left[\frac{1}{2\pi}\left('R_T^{TZ} + b\left(\frac{\partial}{\partial T}{}'\nabla_T^{TZ} - \frac{1}{2}\left['\nabla_T^{TZ}, (g_T^{TZ})^{-1}\frac{\partial}{\partial T}g_T^{TZ}\right]\right)\right)\right]_{b=0}.$$

By (5.59) and (5.60), we get the first equation of (4.30), and we get

$$\begin{aligned}\int_1^{+\infty}\widetilde{e}'_T(TZ)dT &= \widetilde{e}(TZ, '\nabla^{TZ}, '\nabla_\infty^{TZ}) \\ &= \widetilde{e}(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \quad \text{in } Q^W/Q^{W,0}.\end{aligned} \tag{5.61}$$

The proof of Theorem 4.7 is completed. ∎

6. Proof of Theorem 4.5

We use the notation of Section 3.1.

Let $F = F^0 \supset F^1 \supset \cdots \supset F^n = 0$ be a filtration of flat vector bundles of F on S. For $i \geq 0$, set $\mathrm{Gr}^i F = F^i/F^{i+1}$. Let h^F (resp. $h^{\mathrm{Gr}F}$) be Hermitian metrics on F^i(resp. $\mathrm{Gr}\,F$). Let h^{F^i} be the metric on F^i induced by h^F.

Let G^i be the orthogonal sub-bundle of F^{i+1} in F^i. Let h^{G^i} be the metric on G^i induced by $h^{\mathrm{Gr}^i F}$. Let P^{G^i} be the orthogonal projection from F on G^i. We denote N_H the number operator on G^i and $\mathrm{Gr}\,F$. Let $h'^F = \bigoplus h^{G^i}$ be the metric on $F = \bigoplus G^i$.

Let h_T^F $(T \geq 1)$ be a family of metrics on F such that $h_1^F = h^F$ and that there exists $\delta > 0$, such that when $T \to +\infty$, for $s_1 \in F^i, s_2 \in F^j$, we have

$$\begin{aligned}\langle s_1, s_2\rangle_{h_T^F} &= T^{2n-i-j}\Big(\Big\langle P^{G^i}s_1, P^{G^j}s_2\Big\rangle_{h'^F} + O(1/T^\delta)\Big), \\ (h_T^F)^{-1}\frac{\partial}{\partial T}h_T^F &= \frac{1}{T}T^{N_H}\Big(2(n-N_H) + O(1/T^\delta)\Big)T^{-N_H}.\end{aligned} \tag{6.1}$$

PROPOSITION 6.1: *Set*

$$I(h_T^F, h^{\mathrm{GrF}}) = -\int_1^{+\infty} \Big\{\varphi \operatorname{Tr}\Big[(h_T^F)^{-1}(\frac{\partial}{\partial T}h_T^F) f'(\nabla^F, h_T^F)\Big] - 2\varphi \operatorname{Tr}\Big[\frac{n-N_H}{T} f'(\nabla^{\mathrm{GrF}}, h^{\mathrm{GrF}})\Big]\Big\} dT. \tag{6.2}$$

Then

$$I(h_T^F, h^{\mathrm{GrF}}) = T(F, \operatorname{Gr} F, h^F, h^{\mathrm{GrF}}) \quad \text{in } Q^S/Q^{S,0}. \tag{6.3}$$

Proof: For $t \in [0,1]$, set

$$h_{T,1}^F = \bigoplus_i T^{-2i} h^{\mathrm{Gr}^i F}, \quad h_{T,t}^F = t h_{T,1}^F + (1-t) h_T^F. \tag{6.4}$$

By [BLo, (1.24)], as in [BLo, (1.26)], we get

$$\begin{aligned} I(h_T^F, h^{\mathrm{Gr}F}) - I(h_{T,1}^F, h^{\mathrm{Gr}F}) = \\ \varphi d \int_1^{+\infty}\int_0^1 \operatorname{Tr}\Big[(h_T^F)^{-1}(\frac{\partial}{\partial T}h_T^F)(h_{T,t}^F)^{-1}(\frac{\partial}{\partial t}h_{T,t}^F) f''(\omega(F, h_{T,t}^F))\Big] dt dT. \end{aligned} \tag{6.5}$$

By (6.1) and (6.4), we also verify that the right term in (6.5) converges.

The real, even form $I(h_{T,1}^F, h^{\mathrm{GrF}})$ verifies the following conditions as in [BLo, Theorem A1.1]:

(a) The following identity holds,

$$dI(h_{T,1}^F, h^{\mathrm{GrF}}) = f(\nabla^F, h^F) - f(\nabla^{\mathrm{GrF}}, h^{\mathrm{GrF}}). \tag{6.6}$$

(b) If S' is a smooth manifold and $\alpha: S' \to S$ is a smooth map, then

$$I(\alpha^* h_{T,1}^F, \alpha^* h^{\mathrm{GrF}}) = \alpha^* I(h_{T,1}^F, h^{\mathrm{GrF}}). \tag{6.7}$$

(c) If $(F, h^F) = \bigoplus_i (\mathrm{Gr}^i F, h^{\mathrm{Gr}^i \mathrm{F}})$, then $I(h_{T,1}^F, h^{\mathrm{GrF}}) = 0$.

(d) $I(h_{T,1}^F, h^{\mathrm{GrF}})$ depends smoothly on $\nabla^F, \nabla^{\mathrm{GrF}}$ and h^{GrF}.

Now, we can apply the techniques of the proof of [BLo, Theorem A1.2]; we get (6.3) for $h_{T,1}^F$.

By (6.5) and (6.3) for $h_{T,1}^F$, the proof of Proposition 6.1 is completed. ∎

Proof of Theorem 4.5: Let D_T^Z be the operator defined in (1.10) with respect to (π_3, g_T^{TZ}, h^F). Let $\langle\ \rangle_T$ be the metric on E_0 defined in (1.9) with respect to g_T^{TZ}, h^F. Let P_T be the orthogonal projection from E_0 on $\operatorname{Ker} D_T^Z \simeq H(Z, F_{|Z})$ with respect to $\langle\ \rangle_T$. Recall that $h_T^{H(Z,F_{|Z})}$ is the metric on $\operatorname{Ker} D_T^Z \simeq H(Z, F_{|Z})$

induced by $\langle\ \ \rangle_T$. Let $\widetilde{P}_T$ be the orthogonal projection from E_0 on $\widetilde{E}_T = \operatorname{Ker} A_T^{(0)}$ with respcet to $\langle\ \ \rangle$.

By (2.6) and (4.3), the linear map

$$C_T = T^{N_X - \frac{1}{2}\dim X} \colon (E, \langle\ \ \rangle_T, d^Z) \to (E, \langle\ \ \rangle, d_T^Z)$$

is an identification of Hermitian chain complexes. By (2.17), we know

$$P_T = C_T^{-1} \widetilde{P}_T C_T. \tag{6.8}$$

For $\alpha, \alpha' \in E_0$, we have

$$\begin{aligned} \langle P_T\alpha, P_T\alpha'\rangle_T &= \left\langle C_T^{-1}\widetilde{P}_T C_T\alpha, C_T^{-1}\widetilde{P}_T C_T\alpha'\right\rangle_T \\ &= \frac{1}{T^{\dim X}}\left\langle \widetilde{P}_T T^{N_X}\alpha, \widetilde{P}_T T^{N_X}\alpha'\right\rangle. \end{aligned} \tag{6.9}$$

By (5.31) and (5.46), for $n = \dim Z$, we have

$$\widetilde{P}_T = \widetilde{p}_{n+1,T}. \tag{6.10}$$

Now, by Proposition 5.1, and (6.9), (6.10), as in [Ma2, 1(f)], we know the metric $T^{\dim X} h_T^{H(Z,F_{|Z})}$ on $H(Z, F_{|Z})$ verifies the condition (6.1).

Remark also that by [BLo, Proposition 1.3],

$$\begin{aligned} \operatorname{Tr}_s\left[f'\Big(\frac{1}{2}\omega(H(Z,F_{|Z}), h^{H(Z,F_{|Z})})\Big)\right] &= \sum_i (-1)^i \dim H^i(Z, F_{|Z}) \\ &= \chi(Z)\operatorname{rk}\mathrm{F}, \\ \operatorname{Tr}_s\left[f'\Big(\frac{1}{2}\omega(E_\infty, h^{E_\infty})\Big)\right] &= \sum_{p,q}(-1)^{p+q}\dim E_\infty^{p,q} = \chi(Z)\operatorname{rk}\mathrm{F}. \end{aligned} \tag{6.11}$$

By Proposition 6.1 and (6.11), we get Theorem 4.5. ∎

7. Proof of Theorem 4.6

This Section is organized as follows. In Section 7.1, we establish a Lichnerowicz formula for $A^2_{\varepsilon,T/\varepsilon}$. In Section 7.2, by an argument of [BZ, §4], we can use a Getzler rescaling on the operator $L^0_{3,\varepsilon,T}$. Then we prove Theorem 4.6.

In this Section, we use the assumptions and notation of Sections 4 and 5.

7.1. A Lichnerowicz formula. Recall that we denote e_i, f_l, e_a orthonormal bases of (TX, g^{TX}), (TY, g^{TY}), (TZ, g^{TZ}), and that we denote e^i, f^l, e^a the corresponding dual bases. Let g_α be a base of TS and let g^α be the dual base of T^*S. Let $c_T(e_a), \widehat{c}_T(e_a)$ be the action of e_a defined by (1.14) with respect to (TZ, g_T^{TZ}). Then for $u \in \mathbb{R}$,

(7.1)

$$T^{N_X} c_T(Te_i + uf_l) T^{-N_X} = c(e_i + uf_l),\ T^{N_X} \widehat{c}_T(Te_i + uf_l) T^{-N_X} = \widehat{c}(e_i + uf_l).$$

Set

$$L^0_{3,\varepsilon,T} = (T/\varepsilon)^{N_X} C^2_{3,\varepsilon^2,T/\varepsilon} (T/\varepsilon)^{-N_X} = A^2_{\varepsilon,T/\varepsilon}. \tag{7.2}$$

Then

$$\mathrm{Tr}_s\left[*^{-1}_{T/\varepsilon}\Big(\frac{\partial}{\partial T} *_{T/\varepsilon}\Big) g(C^2_{3,\varepsilon^2,T/\varepsilon})\right] = \mathrm{Tr}_s\left[*^{-1}_{T/\varepsilon}\Big(\frac{\partial}{\partial T} *_{T/\varepsilon}\Big) g(L^0_{3,\varepsilon,T})\right]. \tag{7.3}$$

Let $P_{3,\varepsilon,T}(x,x')$ $(x,x' \in Z_s)$ be the smooth kernel associated to the operator $g(L^0_{3,\varepsilon,T})$ calculated with respect to $dv_Z(x')$. For $b \in Y_s$, set

$$g_{\varepsilon,T}(b) = \int_{X_a} \varphi\, \mathrm{Tr}_s\left[*^{-1}_{T/\varepsilon} (\frac{\partial}{\partial T} *_{T/\varepsilon}) P_{3,\varepsilon,T}(x,x)\right] dv_X. \tag{7.4}$$

By (7.3), we get

$$\varphi\, \mathrm{Tr}_s\left[*^{-1}_{T/\varepsilon}(\frac{\partial}{\partial T} *_{T/\varepsilon}) g(C^2_{3,\varepsilon^2,T/\varepsilon})\right] = \int_{Y_s} g_{\varepsilon,T}\, dv_Y. \tag{7.5}$$

Recall that $\nabla^{TX}, \nabla^{TY}, \nabla^{TZ}_T$ are the connections on (TX, g^{TX}), (TY, g^{TY}), (TZ, g^{TZ}_T) defined in Section 1.1. Let R^{TX}, R^{TY}, R^{TZ}_T be the corresponding curvatures. Recall also $\Psi = \omega(F, h^F)$, and let $R^{F,u}$ be the curvature of $\nabla^{F,u}$.

Let ${}'\nabla^{TZ\otimes F,u}_T$ be the connection on $\Lambda(T^*Z) \otimes F$,

$${}'\nabla^{TZ\otimes F,u}_T = T^{N_X} \nabla^{TZ\otimes F,u}_T T^{-N_X}. \tag{7.6}$$

Let $\widehat{R}^{TY} \in \Omega^2(V, \mathrm{End}(\Lambda(T^*Y)), \widehat{R}^{TZ}_T \in \Omega^2(W, \mathrm{End}(\Lambda(T^*Z))$ be defined in (1.18) with respect to (TY, g^{TY}), (TZ, g^{TZ}_T). Then

$$\begin{aligned} T^{N_X} \widehat{R}^{TZ}_T T^{-N_X} =& \frac{1}{4}\Big[\left\langle e_i, R^{TZ}_T e_j\right\rangle_{g^{TX}} \widehat{c}(e_i)\widehat{c}(e_j) \\ &+ \frac{2}{T} \left\langle e_i, R^{TZ}_T f_l\right\rangle_{g^{TX}} \widehat{c}(e_i)\widehat{c}(f_l) + \left\langle f_l, R^{TZ}_T f_m\right\rangle_{g^{TY}} \widehat{c}(f_l)\widehat{c}(f_m)\Big], \end{aligned} \tag{7.7}$$

and define $\mathcal{R}_{3,T} \in \Omega^2(W, \mathrm{End}(\Lambda(T^*Z)\otimes F))$ by

$$\mathcal{R}_{3,T} = (T^{N_X} \widehat{R}^{TZ}_T T^{-N_X} \otimes I_F) + (I_{\Lambda(T^*Z)} \otimes R^{F,u}). \tag{7.8}$$

Let K^Z_T be the scalar curvature of $(TZ, g^{TX} \oplus T^2 \pi_1^* g^{TY})$ along the fibres Z.

By using the Lichnerowicz formula [BLo, Theorem 3.11], we get

$$
\begin{aligned}
(7.9)\qquad L^0_{3,\varepsilon,T} = &-\frac{T^2}{4}\Big({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,e_i} + \frac{T}{\varepsilon^2}\left\langle S_{3,T/\varepsilon}(e_i)e_j, g_\alpha\right\rangle_{T/\varepsilon} c(e_j)g^\alpha \\
&+ \frac{1}{\varepsilon}\left\langle S_{3,T/\varepsilon}(e_i)f_l, g_\alpha\right\rangle_{T/\varepsilon} c(f_l)g^\alpha \\
&+ \frac{1}{\varepsilon^2}\left\langle S_{3,T/\varepsilon}(e_i)g_\alpha, g_\beta\right\rangle_{T/\varepsilon} g^\alpha \wedge g^\beta\Big)^2 \\
&- \frac{\varepsilon^2}{4}\Big({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,f_l} + \frac{1}{\varepsilon}\left\langle S_{3,T/\varepsilon}(f_l)f_m, g_\alpha\right\rangle_{T/\varepsilon} c(f_l)g^\alpha \\
&+ \frac{T}{\varepsilon^2}\left\langle S_{3,T/\varepsilon}(f_l)e_i, g_\alpha\right\rangle_{T/\varepsilon} c(e_i)g^\alpha \\
&+ \frac{1}{\varepsilon^2}\left\langle S_{3,T/\varepsilon}(f_l)g_\alpha, g_\beta\right\rangle_{T/\varepsilon} g^\alpha \wedge g^\beta\Big)^2 \\
&+ \frac{T^2}{16}K^Z_{T/\varepsilon} + \frac{1}{2}g^\alpha \wedge g^\beta \wedge \mathcal{R}_{3,T/\varepsilon}(g_\alpha, g_\beta) \\
&+ \frac{\varepsilon^2}{8}c(f_l)c(f_m)\mathcal{R}_{3,T/\varepsilon}(f_l, f_m) \\
&+ \frac{T^2}{8}c(e_i)c(e_j)\mathcal{R}_{3,T/\varepsilon}(e_i, e_j) + \frac{\varepsilon T}{4}c(e_i)c(f_l)\mathcal{R}_{3,T/\varepsilon}(e_i, f_l) \\
&+ \frac{T}{2}g^\alpha \wedge c(e_i)\mathcal{R}_{3,T/\varepsilon}(g_\alpha, e_i) + \frac{\varepsilon}{2}g^\alpha \wedge c(f_l)\mathcal{R}_{3,T/\varepsilon}(g_\alpha, f_l) \\
&+ \frac{1}{4}\Big\{\frac{T^2}{4}\Psi(e_i)^2 + \frac{\varepsilon^2}{4}\Psi(f_l)^2 + \frac{T^2}{8}\widehat{c}(e_i)\widehat{c}(e_j)[\Psi(e_i), \Psi(e_j)] \\
&+ \frac{\varepsilon T}{4}\widehat{c}(e_i)\widehat{c}(f_l)[\Psi(e_i), \Psi(f_l)] + \frac{\varepsilon^2}{8}\widehat{c}(f_l)\widehat{c}(f_m)[\Psi(f_l), \Psi(f_m)] \\
&- \frac{T^2}{2}c(e_i)\widehat{c}(e_j)({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,e_i}\Psi)(e_j) \\
&- \frac{\varepsilon T}{2}c(e_i)\widehat{c}(f_l)({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,e_i}\Psi)(f_l) \\
&- \frac{\varepsilon T}{2}c(f_l)\widehat{c}(e_i)({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,f_l}\Psi)(e_i) \\
&- \frac{\varepsilon^2}{2}c(f_l)\widehat{c}(f_m)({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,f_l}\Psi)(f_m)\Big\} \\
&- \frac{T}{4}g^\alpha\widehat{c}(e_i)({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,g_\alpha}\Psi)(e_i) - \frac{\varepsilon}{4}g^\alpha\widehat{c}(f_l)({}'\nabla^{TZ\otimes F,u}_{T/\varepsilon,g_\alpha}\Psi)(f_l).
\end{aligned}
$$

7.2. The Getzler rescaling on Y. To calculate the limit as $\varepsilon \to 0$ of $g_{\varepsilon,T}(b)$, we proceed as in [BCh, §4].

First as in [BCh], by using finite propagation speed of solutions of hyperbolic equations [CP, §7.8], [T, §4.4], one can show that the problem calculating the limit of $g_{\varepsilon,T}(b)$ as $\varepsilon \to 0$ is local on Y_s. Namely, if $b_0 \in Y_s$, we may instead assume that Y_s replaced by $(TY)_{b_0} = \mathbf{R}^{m_0}$, with $0 \in (TY)_{b_0} = \mathbf{R}^{m_0}$ representing b_0, and the extended fibration over $\mathbf{R}^{m_0}$ coincides with the given fibration near $0 \simeq b_0$.

Let $\nabla^{\Lambda(T^*X)\otimes F,u}$ be the connection on $\Lambda(T^*X)\otimes F$ induced by $\nabla^{TX}, \nabla^{F,u}$. Let $\nabla^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)}$ be the connection on $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)$ along the fibres Y_s, which is induced by $\nabla^{\Lambda(T^*Y)}$. Recall that S_2 is the tensor defined in Section 1.1 associated to $(\pi_2, T_2^H V, g^{TY})$. Let P^{TY} be the projection from $TV = TY \oplus T_2^H V$ on TY. And the application ψ_ε is defined in (5.35).

Definition 7.1: Let ${}'\nabla^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)}$ be the connection on $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)$ along the fibres Y,

$$(7.10)\quad {}'\nabla_{\cdot}^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)} = \nabla_{\cdot}^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)} + \left\langle S_2(\cdot)f_l, g^H_{\alpha,2}\right\rangle c(f_l)g^\alpha + \left\langle S_2(\cdot)g^H_{\alpha,2}, g^H_{\beta,2}\right\rangle g^\alpha\wedge g^\beta.$$

Let $\nabla^\oplus$ be the connection on

$$\pi_3^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Z)\otimes F \simeq \pi_1^*(\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y))\widehat{\otimes}(\Lambda(T^*X)\otimes F)$$

along the fibres Z,

$$(7.11)\qquad \nabla^\oplus = \pi_1^*{}'\nabla^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)}\otimes 1 + 1\otimes\nabla^{\Lambda(T^*X)\otimes F,u}.$$

For $Y \in \mathbf{R}^{m_0}$, we lift horizontally the paths $t\in\mathbf{R}_+^*\to tY$ into paths $t\in\mathbf{R}_+^*\to x_t\in Z_s$, with $x_t\in Z_{tY}$, $dx/dt\in T^HZ$. For $x_0\in X_0$, we identify TX_{x_t}, $(\pi_3^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Z)\otimes F)_{x_t}$ to TX_{x_0}, $(\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y))_{b_0}\widehat{\otimes}(\Lambda(T^*X)\otimes F)_{x_0}$ by parallel transport along the curve $t\to x_t\in Z_s$ with respect to the connections ∇^{TX}, $\psi_\varepsilon\nabla^\oplus\psi_\varepsilon^{-1}$.

Let Γ be the connection form of ${}'\nabla^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y)}$. By using [ABoP, Proposition 3.7], we see that for $Y\in TY=\mathbf{R}^{m_0}$,

$$(7.12)\qquad \Gamma_Y = \frac{1}{2}({}'\nabla^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y),2})_{b_0}(Y,\cdot) + O(|Y|^2).$$

PROPOSITION 7.1: *The following identity holds,*

$$(7.13)\quad {}'\nabla_{\cdot}^{\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y),2} = \frac{1}{4}\left\langle\nabla^{TY,2}f_l, f_m\right\rangle\Big(c(f_l)c(f_m) - \widehat{c}(f_l)\widehat{c}(f_m)\Big) + \left\langle(S_2P^{TY}S_2 + \nabla^{TY}S_2)g^H_{\alpha,2}, g^H_{\beta,2}\right\rangle g^\alpha\wedge g^\beta + \left\langle(\nabla^{TX}S_2)f_l, g^H_{\alpha,2}\right\rangle c(f_l)g^\alpha.$$

Proof: If $A\in\mathrm{End}(TY)$, the action of A on $\Lambda(T^*Y)$ is given by

$$\frac{1}{4}\sum_{l,m}\langle Af_l, f_m\rangle\Big(c(f_l)c(f_m) - \widehat{c}(f_l)\widehat{c}(f_m)\Big).$$

So we find

$$\nabla^{\Lambda(T^*Y),2} = \frac{1}{4}\sum_{l,m}\langle\nabla^{TY,2}f_l, f_m\rangle\Big(c(f_l)c(f_m) - \widehat{c}(f_l)\widehat{c}(f_m)\Big). \tag{7.14}$$

Now, using the identity

$$[c(f_l)g^\alpha, c(f_m)g^\beta] = 2\delta_{ij}g^\alpha g^\beta,$$

(7.13) follows easily. ∎

By [B4, (11.61)], for $X, Y \in TY$ and $Z, W \in TV$,

$$\begin{aligned}&\langle\nabla^{TY,2}(X,Y)P^{TY}Z, P^{TY}W\rangle + \langle(S_2P^{TY}S_2)(X,Y)Z,W\rangle \\ &\qquad+\langle(\nabla^{TY}S_2)(X,Y)Z,W\rangle = \langle\nabla^{TY,2}(Z,W)X,Y\rangle.\end{aligned} \tag{7.15}$$

The operator $L^0_{3,\varepsilon,T}$ acts on the vector space H_{b_0} of smooth sections of $(\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(T^*Y))_{b_0}\,\widehat{\otimes}(\Lambda(T^*X)\otimes F)_{|X_{b_0}}$ over $\mathbf{R}^{m_0}\times X_{b_0}$.

If $s \in E$, set

$$\begin{aligned}&(F_\varepsilon s)(Y,x) = s(2Y/\varepsilon, x), \quad \text{for } (Y,x)\in\mathbf{R}^{m_0}\times X_{b_0},\\ &L^2_{3,\varepsilon,T} = F_\varepsilon^{-1}L^0_{3,\varepsilon,T}F_\varepsilon.\end{aligned} \tag{7.16}$$

Definition 7.2: For $\varepsilon > 0$, set

$$\widetilde{c}_\varepsilon(f_l) = \frac{2}{\varepsilon}f^l\wedge - \frac{\varepsilon}{2}i_{f_l}. \tag{7.17}$$

Let $L^3_{3,\varepsilon,T}$ be the operator obtained from the operator $L^2_{3,\varepsilon,T}$ by replacing the Clifford variables $c(f_l)$ by the variables $\widetilde{c}_\varepsilon(f_l)$.

Let $P^3_{3,\varepsilon,T}((Y,x),(Y',x'))$ $((Y,x),(Y',x')\in(TY)_{b_0}\times X_{b_0})$ be the smooth kernel associated to the operator $g(L^3_{3,\varepsilon,T})$ with respect to $dv_{(TY)_{b_0}}(Y')dv_{X_{b_0}}(x')$.

If $x\in X_{b_0}$, we can write

$$\begin{aligned}&\frac{1}{T}(2N_X - \dim X)P^3_{3,\varepsilon,T}((0,x),(0,x)) = \\ &\quad\sum_{\substack{1\le i_1<\cdots<i_p\le m_0\\ 1\le j_1<\cdots<j_q\le m_0\\ 1\le k_1<\cdots<k_r\le m_0}} f^{i_1}\wedge\cdots\wedge f^{i_p}\wedge i_{f_{j_1}}\cdots\wedge i_{f_{j_q}}\widehat{c}(f_{k_1})\cdots\widehat{c}(f_{k_r})\widehat{\otimes}R^{i_1\cdots i_p;j_1\cdots j_q;k_1\cdots k_r}_{\varepsilon,T},\end{aligned} \tag{7.18}$$

with $R^{i_1\cdots i_p;j_1\cdots j_q;k_1\cdots k_r}_{\varepsilon,T}\in\pi_2^*\Lambda(T^*S)\widehat{\otimes}\operatorname{End}(\Lambda(T^*X)\otimes F)$.

Set

$$\Big[\frac{1}{T}(2N_X-\dim X)P^3_{3,\varepsilon,T}((0,x),(0,x))\Big]^{\max} = R^{1,\cdots,m_0;;1,\cdots,m_0}_{\varepsilon,T}((0,x),(0,x)). \tag{7.19}$$

By [BZ, Proposition 4.9], among the monomials in the $c(f_l), \widehat{c}(f_l)$'s, only $c(f_1)\widehat{c}(f_1)\cdots c(f_{m_0})\widehat{c}(f_{m_0})$ has a nonzero supertrace. Moreover,

$$\operatorname{Tr}_s[c(f_1)\widehat{c}(f_1)\cdots c(f_{m_0})\widehat{c}(f_{m_0})] = (-2)^{m_0}.$$

By proceeding as in [BL, Proposition 11.2] (consider $\varepsilon/2$ as u there), we get

$$\begin{aligned} &\varphi \operatorname{Tr}_s\left[\frac{1}{T}(2N_X - \dim X)P_{3,\varepsilon,T}((0,x),(0,x))\right] \\ &= 2^{m_0}(-1)^{\frac{m_0(m_0+1)}{2}}\varphi\operatorname{Tr}_s\left[\left[\frac{1}{T}(2N_X-\dim X)P^3_{3,\varepsilon,T}((0,x),(0,x))\right]^{\max}\right]. \end{aligned} \tag{7.20}$$

We denote ∇_{f_l} the ordinary differential operator on $(TY)_{b_0}$ in the direction f_l. By using Theorems 5.1 and 5.2, (5.17), (7.9) and (7.12)–(7.14), and proceeding as in [Ma1, §7], we get, when $\varepsilon \to 0$,

$$L^3_{3,\varepsilon,T} \to L^3_{3,0,T} = -\left(\nabla_{f_l} + \frac{1}{4}\langle R^{TY}_{b_0}Y, f_l\rangle_{g^{TY}_{b_0}}\right)^2 + \widehat{R}^{TY}_{b_0} + C^2_{1,T^2}. \tag{7.21}$$

7.3. Proof of Theorem 4.6. Set

$$H^{TY}(Y) = -\left(\nabla_{f_l} + \frac{1}{4}\langle R^{TY}_{b_0}Y, f_l\rangle_{g^{TY}_{b_0}}\right)^2 + \widehat{R}^{TY}_{b_0}. \tag{7.22}$$

For $Y, Y' \in (TY)_{b_0}$, let $p_t(Y,Y')$ be the smooth kernel associated to the operator $\exp(tH^{TY})$ calculated with respect to $dv_{(TY)_{y_0}}(Y')$.

Let $q_{T^2}(x,x'), q_{f,T^2}(x,x')$ $(x, x' \in X)$ be the smooth kernel associated to the operator $\exp(-C^2_{1,T^2})$, $f'(D_{1,T^2}) = (1-2C^2_{1,T^2})\exp(-C^2_{1,T^2})$ with respect to $dv_X(x')$. Using (7.21), and proceeding as in [B1, §5], we see that for $x \in X_{b_0}$, as $\varepsilon \to 0$,

$$\begin{aligned} P^3_{3,\varepsilon,T}((0,x),(0,x)) \to & p_1(0,0)q_{f,T^2}(x,x) \\ & - 2(H^{TY}(Y)p_1(Y,Y'))_{|Y=Y'=0}\,q_{T^2}(x,x). \end{aligned} \tag{7.23}$$

If $R \in \Lambda(T^*V)\widehat{\otimes}\widehat{c}(TY)$, then there exist $R^{l_1\cdots l_q} \in \Lambda(T^*V)$ such that

$$R = \sum_{1\le l_1<\cdots<l_q\le m_0} R^{l_1\cdots l_q}\widehat{c}(f_{l_1})\cdots\widehat{c}(f_{l_q}). \tag{7.24}$$

We denote

$$\{R\}^{\widehat{c}} = R^{1,\dots,m_0}. \tag{7.25}$$

By [BeGeV, §4.2], we get

$$
\begin{aligned}
\{p_t(0,0)\}^{\widehat{c}} &= \Big\{(4\pi t)^{-m_0/2}\det{}^{1/2}\Big(\frac{tR^{TY}/2}{\sinh(tR^{TY}/2)}\Big)\exp(-t\widehat{R}^{TY})\Big\}^{\widehat{c}} \\
&= 2^{-m_0}(-1)^{\frac{m_0(m_0+1)}{2}} e(TY,\nabla^{TY}), \qquad (7.26)\\
\{(H^{TY}(Y)p_1(Y,Y'))_{|Y=Y'=0}\}^{\widehat{c}} &= -\frac{\partial}{\partial t}\{p_t(0,0)\}^{\widehat{c}}_{|t=1} = 0.
\end{aligned}
$$

By (4.18), (7.20), (7.21), (7.23) and (7.26), we get that, as $\varepsilon \to 0$,

$$
\begin{aligned}
(7.27)\quad \lim_{\varepsilon\to 0}\varphi\,\mathrm{Tr}_s\Big[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})f'(D_{3,\varepsilon^2,T/\varepsilon})\Big] \\
= \frac{1}{T}\int_Y e(TY,\nabla^{TY})\varphi\,\mathrm{Tr}_s\Big[(2N_X-\dim X)f'(D_{1,T^2})\Big].
\end{aligned}
$$

To finish the proof of Theorem 4.6, we must calculate $\mathrm{Tr}_s[f'(D_{1,T^2})]$. At first, as in [BLo, (3.74)],

$$
(7.28)\qquad \frac{\partial}{\partial T}\mathrm{Tr}_s[f'(D_{1,T^2})] = 0.
$$

By using local index theory as in [BLo, p. 334], we get

$$
(7.29)\qquad \lim_{T\to 0}\varphi\,\mathrm{Tr}_s[f'(D_{1,T^2})] = \int_X e(TX,\nabla^{TX})\,\mathrm{Tr}[(1-2R^{F,u})\exp(-R^{F,u})].
$$

By [BLo, Proposition 1.3],

$$
(7.30)\qquad \mathrm{Tr}[(1-2R^{F,u})\exp(-R^{F,u})] = \mathrm{rk}(F).
$$

So, for $T>0$,

$$
(7.31)\qquad \mathrm{Tr}_s[f'(D_{1,T^2})] = \mathrm{rk}(F)\chi(X).
$$

By (7.27) and (7.31), the proof of Theorem 4.6 is completed. ∎

8. Proof of Theorem 4.8

This Section is organized as follows. In Section 8.1, we state a result from which Theorem 4.8 immediately follows. In Section 8.2, using finite propagation speed, we show that the proof of our main result is local on Y_s. In Section 8.3, we prove our main result.

Here, we use the assumptions and notation in Sections 4.2 and 7.

8.1. A REFORMULATION OF THEOREM 4.8.

THEOREM 8.1: *There exists $C > 0$ such that for $0 < u \leq 1$, $T \geq 1$,*

$$\left| \mathrm{Tr}_s \left[*_T^{-1} \frac{\partial}{\partial T}(*_T) f'(D_{3,u^2/T^2,T}) \right] - 2 \int_Z \tilde{e}'_T(TZ) f(\nabla^F, h^F) \right| \leq \frac{Cu}{T}. \tag{8.1}$$

Remark 8.1: Theorem 8.1 implies Theorem 4.8. In fact, for $0 < \varepsilon \leq 1$, $\varepsilon \leq T \leq 1$ we use (8.1), with $u = T$ and T replaced by T/ε, then we find that the right-hand side of (8.1) is dominated by $C\varepsilon$. So we have proved (4.31).

8.2. LOCALIZATION OF THE PROBLEM. Let $r > 0$ such that

$$r < \inf_{s \in S} \{\text{injective radius of the fiber } (Y_s, g^{TY})\}.$$

Let $\alpha \in]0, r/4]$. For $b \in V$, let $B^Y(b, \alpha)$ be the open ball of center b and radius α.

Let f_1 be a smooth even function defined on $\mathbf{R}$ with values in $[0, 1]$, such that

$$\begin{aligned} f_1(t) =&1 \quad \text{for } |t| \leq \alpha/2, \\ &0 \quad \text{for } |t| \geq \alpha. \end{aligned} \tag{8.2}$$

Set

$$g_1(t) = 1 - f_1(t). \tag{8.3}$$

Definition 8.1: For $u \in]0, 1]$, $a \in \mathbf{C}$, set

$$\begin{aligned} F_u(a) &= \int_{-\infty}^{+\infty} (1 - 2a^2) \exp(ita\sqrt{2}) \exp(-t^2/2) f_1(ut) \frac{dt}{\sqrt{2\pi}}, \\ G_u(a) &= \int_{-\infty}^{+\infty} (1 - 2a^2) \exp(ita\sqrt{2}) \exp(-t^2/2) g_1(ut) \frac{dt}{\sqrt{2\pi}}. \end{aligned} \tag{8.4}$$

Clearly

$$F_u(a) + G_u(a) = (1 - 2a^2) \exp(-a^2). \tag{8.5}$$

The functions $F_u(a), G_u(a)$ are even holomorphic functions. So there exist holomorphic functions $\widetilde{F}_u(a)$, $\widetilde{G}_u(a)$ such that

$$F_u(a) = \widetilde{F}_u(a^2), \quad G_u(a) = \widetilde{G}_u(a^2). \tag{8.6}$$

The restrictions of $F_u, G_u, \widetilde{F}_u, \widetilde{G}_u$ to $\mathbf{R}$ lie in the Schwartz space $S(\mathbf{R})$.

PROPOSITION 8.1: *There exist $c > 0, C > 0$ such that for $0 < u \leq 1$, $T \geq 1$,*

$$\left| \operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} \widetilde{G}_{u/T}(C^2_{3,u^2/T^2,T}) \right] \right| \leq C \exp\Big(-c \frac{T^2}{u^2} \Big). \tag{8.7}$$

Proof: By using (8.4), and proceeding as in the proof of [Ma1, Proposition 8.4], we get (8.7). ∎

By Proposition 8.1, and the argument in Section 7.2, the proof of (8.1) can be localized on Y_s.

For $b_0 \in Y_s$, we replace Z_s by $\mathbf{R}^{m_0} \times X_{b_0}$ as in Section 7.2. We also trivialize the fibres as in Section 7.2. Then we will prove (8.1) in this situation.

8.3. PROOF OF THEOREM 8.1. For $n \in \mathbf{N}$, we denote $[n/2] \in \mathbf{Z}$ such that $[n/2] \in]n/2 - 1, n/2]$. Let $\{n/2\} = n/2 - [n/2]$.

By (1.26), (4.18) and (5.35),

$$\varphi \operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} f'(D_{3,u^2/T^2,T}) \right] = \psi_u \varphi \operatorname{Tr}_s \left[\frac{1}{T} (2N_X - \dim X) g(u^2 L^0_{3,\frac{1}{T},1}) \right]. \tag{8.8}$$

By (7.21), (8.8), and standard results on heat kernels, we know that there exist $C > 0$, and some $\mathcal{C}^\infty$ forms $a_{T,j}$ ($j \geq -n, n = \dim Z$) on S, which depend continuously on $T \in [1, +\infty]$, such that for $u \in]0,1]$, $T \in [1, +\infty[$,

$$\left| \operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} f'(D_{3,u^2/T^2,T}) \right] - \sum_{j=-[n/2]}^{0} \frac{1}{T} a_{T,j} u^{2j - 2\{n/2\}} \right| \leq C \frac{u}{T}. \tag{8.9}$$

THEOREM 8.2: *For $T \geq 1$ fixed, when $u \to 0$, we get*

(8.10)

$$\operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} f'(D_{3,u^2,T}) \right] = \begin{cases} 2 \int_Z \widetilde{e}'_T(TZ) f(\nabla^F, h^F) + O(u^2), & \text{if } \dim Z \text{ is even}, \\ O(u), & \text{if } \dim Z \text{ is odd}. \end{cases}$$

Proof: We use the same notation as in Section 1.1 and (4.26)–(4.28). Clearly, $(\rho^* F, \rho^* \nabla^F)$ is a flat vector bundle on $\widehat{W}$.

Using the product structure on $\widehat{W}$, we can write

$$d^{\widehat{W}} = d^W + dT \frac{\partial}{\partial T}, \quad (d^{\widehat{W}})^* = (d^W)^*_T + dT \Big(\frac{\partial}{\partial T} + *_T^{-1} \frac{\partial *_T}{\partial T} \Big). \tag{8.11}$$

Defining $\widehat{C}_{3,u}, \widehat{D}_{3,u}$ as in (1.11) with respect to $(\widehat{\pi}_3, g^{T\widehat{Z}})$, we have

$$\widehat{D}_{3,u} = T^{-N_Z} D_{3,u,T} T^{N_Z} + \frac{1}{2} dT *_T^{-1} \frac{\partial *_T}{\partial T}. \tag{8.12}$$

We deduce that

$$f(\widehat{C}'_{3,u^2}, h^{\widehat{E}}) = f(C'_{3,u^2,T}, h^E_T) + \frac{1}{2} dT \varphi \operatorname{Tr}_s \left[*_T^{-1} \frac{\partial *_T}{\partial T} f'(D_{3,u^2,T}) \right]. \tag{8.13}$$

By Theorem 1.1, comparing the dT term of (8.13), we get Theorem 8.2. ∎

Compare (8.9) and (8.10); we know

$$a_{T,j} = 0 \quad \text{if } j < 0, \qquad \frac{1}{T} a_{T,0} = 2 \int_Z \widetilde{e}'_T(TZ) f(\nabla^F, h^F). \tag{8.14}$$

By (8.9) and (8.14), we get (8.1). ∎

9. Proof of Theorem 4.9

We use the same notation as in Section 8.1.

PROPOSITION 9.1: *There exists $C > 0$ such that for $0 < \varepsilon \leq 1$, $T \geq 1$,*

$$\Big|\varphi \operatorname{Tr}_s[(2N_X - \dim X)\widetilde{G}_\varepsilon(C^2_{3,\varepsilon^2,T/\varepsilon})] - \varphi \operatorname{Tr}_s[(2N_X - \dim X)\widetilde{G}_\varepsilon(C^2_{2,\varepsilon^2})]\Big| \leq \frac{C}{T^\delta} \exp\Big(-\frac{C}{\varepsilon^2}\Big). \tag{9.1}$$

Proof: By using Theorem 5.3, the proof is essentially as same as in [Ma1, Proposition 9.1]. ∎

Using (8.5) and (9.1), it is clear that to establish Theorem 4.9, we only need to establish the following result,

THEOREM 9.1: *If $\alpha > 0$ is small enough, there exist $\delta > 0, C > 0$, such that for $0 < \varepsilon \leq 1$, $T \geq 1$,*

$$\Big|\varphi \operatorname{Tr}_s[(2N_X - \dim X)\widetilde{F}_\varepsilon(C^2_{3,\varepsilon^2,T/\varepsilon})] - \varphi \operatorname{Tr}_s[(2N_X - \dim X)\widetilde{F}_\varepsilon(C^2_{2,\varepsilon^2})]\Big| \leq \frac{C}{T^\delta}. \tag{9.2}$$

By (5.10),

$$\varphi \operatorname{Tr}_s[(2N_X - \dim X)\widetilde{F}_\varepsilon(C^2_{3,\varepsilon^2,T/\varepsilon})] = \varphi \operatorname{Tr}_s[(2N_X - \dim X)\widetilde{F}_\varepsilon(A^2_{\varepsilon,T/\varepsilon})]. \tag{9.3}$$

Let $\widetilde{F}_\varepsilon(A^2_{\varepsilon,T/\varepsilon})(x,x')$ $(x, x' \in Z_s)$ be the smooth kernel associated to the operator $\widetilde{F}_\varepsilon(A^2_{\varepsilon,T/\varepsilon})$ with respect to $dv_Z(x')$. Using finite propagation speed of solutions of hyperbolic equations [CP, §7.8], [T, §4.4], we know the proof of (9.2) is local on Y. As in [BerB, §8b)], there is also a smooth $\mathbf{Z}$-graded vector bundle $K \subset \Omega_{b_0} = \mathcal{C}^\infty(X_{b_0}, \Lambda(T^*X) \otimes F)$ over $(TY)_{b_0} \simeq \mathbf{R}^{m_0}$ which coincides with $\operatorname{Ker} D^X$ on $B^Y(0,2\alpha_0)$ $(0<\alpha_0<\alpha)$, with $\operatorname{Ker} D^X_{b_0}$ over $TY\backslash B^Y(0,3\alpha_0)$, and such that if $K^\perp$ is the orthogonal bundle to K in Ω_{b_0},

$$K^\perp \cap \operatorname{Ker} D^X_{b_0} = \{0\}. \tag{9.4}$$

Let P_Y $(Y \in \mathbf{R}^{m_0})$ be the orthogonal projection operator from Ω_{b_0} on K_Y. Set $P_Y^\perp = 1 - P_Y$.

Let $\psi\colon \mathbf{R} \to [0,1]$ be a smooth function such that

$$\begin{aligned} \psi(t) =& 1 \quad \text{for } |t| \le \alpha_0, \\ =& 0 \quad \text{for } |t| \ge 2\alpha_0. \end{aligned} \tag{9.5}$$

Let Δ^{TY} be the standard Laplacian on $(TY)_{b_0}$ with respect to the metric $h^{TY_{|b_0}}$. Let H_{b_0} be the vector space of smooth sections of

$$\pi_1^*(\pi_2^*(\Lambda(T^*S))\widehat{\otimes}\Lambda(T^*Y))_{b_0}\widehat{\otimes}(\Lambda(T^*X)\otimes F)_{|X_{b_0}}$$

over $(TY)_{b_0} \times X_{b_0}$. Let $L^1_{\varepsilon,T}$ be the operator

$$L^1_{\varepsilon,T} = \psi^2(|Y|)A^2_{\varepsilon,T/\varepsilon} + \frac{1}{4}(1-\psi^2(|Y|))\Big(-\varepsilon^2\Delta^{TY} + T^2 P_Y^\perp D^{X,2}_{b_0} P_Y^\perp\Big). \tag{9.6}$$

Recall that the operation F_ε is defined in (7.16),

$$L^2_{\varepsilon,T} = F_\varepsilon^{-1} L^1_{\varepsilon,T} F_\varepsilon. \tag{9.7}$$

Let $\mathcal{O}_p$ be the set of differential operators acting on smooth sections of $(\Lambda(T^*X)\otimes F)_{X_{b_0}}$ over $\mathbf{R}^{m_0} \times X_{b_0}$. Then we find that

$$L^2_{\varepsilon,T} \in \pi_2^*(\Lambda(T^*S))\widehat{\otimes}\operatorname{End}(\Lambda(T^*Y))_{b_0}\widehat{\otimes}\mathcal{O}_p.$$

Let $L^3_{\varepsilon,T}$ be obtained from $L^2_{\varepsilon,T}$ by replacing the Clifford variables $c(f_j)$ $(1 \le j \le m_0)$ by the operators $c_\varepsilon(f_j)$ as in Section 7.

Let $\hat{\Lambda}(T^*Y)$ be another copy of $\Lambda(T^*Y)$, and $\hat{c}(U)$ $(U \in TY)$ acts on it. Let $\mathbf{E}^0$ be the vector space of square integrable sections of

$$\pi_2^*(\Lambda(T^*S))\widehat{\otimes}\Lambda(T^*Y)\widehat{\otimes}\hat{\Lambda}(T^*Y)\widehat{\otimes}\Lambda(T^*X)\otimes F$$

over $(TY)_{b_0} \times X_{b_0}$. Let $\mathbf{F}^0_\varepsilon$ be the vector space of square integrable sections of

$$\pi_2^*(\Lambda(T^*S))\widehat{\otimes}\Lambda(T^*Y)\widehat{\otimes}\hat{\Lambda}(T^*Y)\widehat{\otimes}S_\varepsilon^{-1*}K$$

over $(TY)_{b_0}$. Then $\mathbf{F}^0_\varepsilon$ is a Hilbert subspace of $\mathbf{E}^0$. Let $\mathbf{F}^{0,\perp}_\varepsilon$ be its orthogonal complement in $\mathbf{E}^0$. Let p_ε be the orthogonal projection operator from $\mathbf{E}^0$ on $\mathbf{F}^0_\varepsilon$; set $p_\varepsilon^\perp = 1 - p_\varepsilon$. Then if $s \in \mathbf{E}^0$,

$$p_\varepsilon s(Y) = P_{\varepsilon Y} s(Y,\cdot) \quad \text{for } Y \in (TY)_{b_0}. \tag{9.8}$$

Put

$$E_{\varepsilon,T} = p_\varepsilon L^3_{\varepsilon,T} p_\varepsilon, \quad F_{\varepsilon,T} = p_\varepsilon L^3_{\varepsilon,T} p_\varepsilon^\perp, \tag{9.9}$$
$$G_{\varepsilon,T} = p_\varepsilon^\perp L^3_{\varepsilon,T} p_\varepsilon, \quad H_{\varepsilon,T} = p_\varepsilon^\perp L^3_{\varepsilon,T} p_\varepsilon^\perp.$$

Then we write $L^3_{\varepsilon,T}$ in matrix form with respect to the splitting $\mathbf{E}^0 = \mathbf{F}^0_\varepsilon \oplus \mathbf{F}^{0,\perp}_\varepsilon$,

$$L^3_{\varepsilon,T} = \begin{bmatrix} E_{\varepsilon,T}, & F_{\varepsilon,T} \\ G_{\varepsilon,T}, & H_{\varepsilon,T} \end{bmatrix}. \tag{9.10}$$

By using (7.9), we have

THEOREM 9.2: *There exist operators $E_\varepsilon, F_\varepsilon, G_\varepsilon, H_\varepsilon$ such that as $T \to +\infty$,*

$$E_{\varepsilon,T} = E_\varepsilon + O(1/T), \quad F_{\varepsilon,T} = TF_\varepsilon + O(1), \tag{9.11}$$
$$G_{\varepsilon,T} = TG_\varepsilon + O(1), \quad H_{\varepsilon,T} = T^2 H_\varepsilon + O(T).$$

Let Q_ε be the operator obtained from $\frac{1}{2}\psi^2(Y)[D^X, \mathcal{C}]$ by proceeding as before, i.e., by rescaling the coordinate Y and the Clifford variables $c(f_l)$. Then Q_ε is the first order elliptic operator along the fibres X, and $Q_\varepsilon(\mathbf{F}^0_\varepsilon) \subset \mathbf{F}^{0,\perp}_\varepsilon$, and

$$F_\varepsilon = p_\varepsilon Q_\varepsilon p_\varepsilon^\perp, \quad G_\varepsilon = p_\varepsilon^\perp Q_\varepsilon p_\varepsilon, \tag{9.12}$$
$$H_\varepsilon = \frac{1}{4} p_\varepsilon^\perp \Big(\varphi^2(\varepsilon|Y|) D^{X,2}_{\varepsilon Y} + (1 - \varphi^2(\varepsilon|Y|)) D^{X,2}_{b_0}\Big) p_\varepsilon^\perp.$$

We can now apply the techniques and results of [Ma1, §9] to complete the proof of Theorem 9.1. In fact, (8.4) allows us to localize the problem on Y_s as in Section 7.

It is easy to see that the last terms of (7.9) don't cause any trouble in modifying the estimates in [Ma1, §9], so that all the arguments can actually go through here. By proceeding in exactly the same way as in [Ma1, §9], we get Theorem 9.1. ∎

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